MAE/MSE 502, Spring 2017, Homework #3

Prob. 1 (4 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, consider the PDE

$$\frac{\partial u}{\partial t} = U \frac{\partial u}{\partial x} + K \frac{\partial^2 u}{\partial x^2} \quad \text{(where } U \text{ and } K \text{ are constants)} ,$$

with periodic boundary conditions in the x-direction (i.e., $u(0, t) = u(2\pi, t)$, $u_x(0, t) = u_x(2\pi, t)$, and so on), and the boundary condition in t-direction given as

$$u(x, 0) = \mathrm{P}(x) .$$

(a) Solve the PDE by Fourier series expansion. You may express the solution as an infinite series. If $\mathrm{P}(x)$ is given as $\mathrm{P}(x) \equiv \exp([1-\cos(x)]^2)$, plot the solution at $t = 1$ for the three cases with (i) $U = 1$, $K = 0$, (ii) $U = -1$, $K = 0$, and (iii) $U = 1$, $K = 0.3$. Also, plot the solution at $t = 0$ (which is the same for all three cases). Please collect all four curves in one plot. [To make the plot, the Fourier series needs to be truncated. See the general remark below HW1-Q1 on estimating the appropriate number of terms to keep in the series.]

(b) Show that, when $K = 0$, the solution of the system is $u(x, t) = \mathrm{P}(x+Ut)$ for any given $\mathrm{P}(x)$. In other words, the "initial structure" $\mathrm{P}(x)$ never changes its shape but simply drifts as a whole at a constant "speed" of $-U$ in the x-direction. (For example, if $U = 3$, it will drift by 1.5 unit *to the left* at $t = 0.5$.) [Note: Although the proof can be constructed by several different methods, including the Method of Characteristics which we will discuss later, <u>for this particular exercise we ask that the proof be given based the solution of the PDE by Fourier series expansion</u>.]

Prob. 2 (3 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$(1+t) \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + u \quad ,$$

with periodic boundary conditions in the x-direction, and the boundary condition in the t-direction given as

$$u(x, 0) = 1 + \sin(x) .$$

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals. The solution should be real and should be expressed all in real numbers and functions.

Prob 3 (2 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 + t + t\cos(\pi x) \quad ,$$

with the boundary conditions

(i) $u_x(0, t) = 0$
(ii) $u_x(1, t) = 0$
(iii) $u(x, 0) = 3 + \cos(\pi x) + \cos(2\pi x)$.

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals.

Prob 4 (2 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^3 u}{\partial t^3} = \frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} + 1 \quad ,$$

with periodic boundary conditions in the x-direction and the following boundary conditions in the t-direction,

(i) $u(x, 0) = \cos(x)$,
(ii) $u_t(x, 0) = 1$,
(iii) $u_{tt}(x, 0) = \sin(x)$

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals. The solution should be real and should be expressed in real numbers and functions.