MAE/MSE 502, Spring 2017, Homework #4

You might find the following formula useful for this homework:

$$\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2} \qquad \textbf{Eq. (1)}$$

Prob 1 (3 points)
For $u(x,t)$ defined on the infinite domain of $-\infty < x < \infty$ and $t \geq 0$, use the Fourier transform method to solve the PDE,

$$\frac{\partial u}{\partial t} = A\,\frac{\partial^3 u}{\partial x^3} + B\,\frac{\partial^5 u}{\partial x^5} \quad ,$$

with the boundary conditions

(i) $u(x, t)$ and its partial derivatives with respect to x vanish as $x \to \pm\infty$
(ii) $u(x,0) = \exp(-x^2)$.

It is acceptable to express the solution as a integral. Consider the following 3 cases:
(a) With $A = 1$, $B = 0$, plot $u(x, t)$ as a function of x at $t = 0$, and 0.5. Please collect the two curves in the same plot. Please make the plot over the range of $-20 \leq x \leq 20$.
(b) Repeat (a) but with $A = 0$, $B = 1$.
(c) Repeat (a) but with $A = 1$, $B = 1$.
Briefly explain the behavior of the solution in those 3 cases.

[Note: For this problem, numerical integration (e.g., by the trapezoidal method) will be needed to evaluate $u(x, t)$ in order to make the plots. Since numerical integration cannot go all the way to ∞, one has to "truncate" the integral at a finite value of ω. This is analogous to truncating a Fourier series at a finite n. A useful way to determine where to truncate the integral is to plot, for a give t, $U(\omega, t)$ (the Fourier transform of $u(x, t)$) as a function of ω and observe how $U(\omega, t)$ decays with ω.]

Prob 2 (3 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, use the method of Fourier transform to solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \quad ,$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives with respect to x vanish as $x \to \pm\infty$
(ii) $u(x,0) = \exp(-x^2)$
(iii) $u_t(x,0) = 0$

To receive full credit, the final solution should have a closed-form expression of a *real* function that contains no unevaluated integrals. Plot the solution $u(x,t)$ as a function of x at $t = 0$, 0.5, and 1. Please collect all three curves in one plot.

Prob 3 (2 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \quad ,$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives with respect to x vanish as $x \to \pm \infty$
(ii) $u(x,0) = \exp(-x^2)$

To receive full credit, the final solution should have a closed-form expression of a *real* function that contains no unevaluated integrals.

Prob 4 (2 points)
Consider the following ODE for $u(t)$ with $t \geq 0$,

$$\frac{du}{dt} = u \ \ln(t+2) + Q(t) \quad \text{(where "ln" is natural log)} \ .$$

Find the Green's function, $G(t, t')$, such that for any given $Q(t)$ and initial condition $u(0)$ the solution of the system can be expressed as

$$u(t) = G(t,0)u(0) + \int_0^t G(t,t')Q(t')dt' \quad .$$

Hint: For Prob 2 and 3, the integral formula in Eq. (1) can be used twice to process both Fourier transform and inverse Fourier transform. For Prob 1, it can be used to process Fourier transform while numerical integration is needed to complete the inverse transform.