MAE502, Fall 2018 Homework # 2

Hard copy of report is due 6:00 PM on the due date. Computer codes used to complete the tasks should be included in the report.

Task 0 (no point, but mandatory to complete for the report to be accepted)
Provide a statement to address whether collaboration occurred in completing this assignment. **This statement must be placed in the beginning of the first page of report.** See related clarifications in Homework #1.

Task 1 (2.5 points)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \, ,$$

with the boundary conditions,

(i) $u(0, y) = 0$ (ii) $u(1, y) = 4\,(\sqrt{y} - y)$ (iii) $u(x, 0) = 0$ (iv) $u(x, 1) = \sin(\pi\,\sqrt{x}\,)$.

Make a contour plot of the solution, $u(x, y)$. See additional note in the last page for an example of using Matlab to make a contour plot. For this problem, we expect the solution to be expressed as an infinite series. Please see the remark below HW1-Task1 on how to truncate the series and numerically compute the expansion coefficients.

Task 2 (3 points)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + 4\,\frac{\partial^2 u}{\partial y^2} + 4\,\pi^2 u = 0 \, ,$$

with the boundary conditions (u_x and u_y denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively),

(i) $u_x(0, y) = 0$ (ii) $u_x(1, y) = 0$ (iii) $u(x, 0) = 1$ (iv) $u_y(x, 1) = 2 + \cos(2\pi x) + \cos(3\pi x)$.

(Note that the 3^{rd} boundary condition is imposed on u, while all other boundary conditions are imposed on the derivative of u.) Make a contour plot of the solution, $u(x, y)$. For this problem, we expect a closed-form analytic solution that consists of only a finite number of terms and without any unevaluated integral. A deduction will be assessed on the solution otherwise.

Task 3 (3 point)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ ,$$

with the boundary conditions (u_x and u_y denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively),

(i) $u_x(0, y) = 0$ (ii) $u_x(1, y) = 0$ (iii) $u_y(x, 0) = 1$ (iv) $u_y(x, 1) = 1 + \cos(\pi x)$

(a) Test the solvability condition on the system. Based on it, which of the following is true?
(I) The system has no solution. (II) The system has a unique solution. (III) The system has multiple solutions. If your answer is (I), no need to proceed further. Otherwise, proceed to Part (b).
(b) Find the solution(s) of the system. If your answer for Part (a) is (III), please clearly write out what those multiple solutions are.
(c) Repeat the exercise in (a) (and (b), if necessary) but now consider a modified system in which the third boundary condition is changed to: (iii) $u_y(x, 0) = 0$.

Task 4 (0.5 point)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, consider the system of *Poisson equation* (where the "source term" Q is a given function of x and y),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = Q \ ,$$

with the boundary conditions (u_x and u_y denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively),

(i) $u_x(0, y) = 0$ (ii) $u_x(1, y) = 0$ (iii) $u_y(x, 0) = 0$ (iv) $u_y(x, 1) = 0$.

Let $Q(x,y) = x + y^2 + K$ where K is a constant. For what value(s) of K may a solution exist for the system?

Task 5 (2 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 6$ and $t \ge 0$, solve the Wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ ,$$

with the boundary conditions (u_x and u_t denote $\partial u/\partial x$ and $\partial u/\partial t$, respectively),

(i) $u(0, t) = 0$ (ii) $u(6, t) = 0$ (iii) $u(x, 0) = \mathrm{P}(x)$ (iv) $u_t(x, 0) = 0$,

where

$$\mathrm{P}(x) = \begin{cases} x & , \text{ if } x \le 2 \\ (6 - x)/2 & , \text{ if } 2 < x \le 6 \ . \end{cases}$$

Plot the solution $u(x, t)$ as a function of x at t = 0, 1.8, 3.0, 4.2, 6.0, and 10.8. Please collect all 6 curves in one plot.

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x)\exp(-2y)$ over the domain of $0 \le x \le 1$ and $0 \le y \le 1$, using the contour levels of (-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9). The contours for u = -0.7, -0.3, 0.3, and 0.7 are labeled. It is essential to provide the coordinates of the grid (x2d and y2d in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y. A black-and-white contour plot is acceptable as long as the contours are properly labeled.

```
clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u,[-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h,[-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')
```