# MAE502, Fall 2018 Homework # 3

Hard copy of report is due 6:00 PM on the due date. Computer codes used to complete the tasks should be included in the report.

**Task 0** (no point, but mandatory to complete for the report to be accepted)
Provide a statement to address whether collaboration occurred in completing this assignment. **This statement must be placed in the beginning of the first page of report.** See related clarifications in Homework #1.

**Task 1** (2 points)
Consider the eigenvalue problem for $G(x)$ defined on the interval, $0 \leq x \leq 5$,

$$\frac{d^2 G}{dx^2} = c\, G$$

with the boundary conditions,

$G'(0) = 1$ , $G(5) = 0$ (Be aware that the 1st b.c. is imposed on the derivative of $G$.)

**(a)** Determine the eigenvalues and the corresponding eigenfunctions of this problem. Are the eigenvalues discrete? For example, if the boundary conditions are replaced by the familiar $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2 \pi^2$ ($n$ is an integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values of $c$ within an interval are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

**(b)** Plot the eigenfunctions, $G_c(x)$, associated with the eigenvalues $c = -1, -0.3, 0, 0.3$, and $1$. (You will find in Part (a) that all those values are indeed valid eigenvalues.) Please collect all 5 curves in a single plot. Note that $G(x)$ is defined only on the interval of $0 \leq x \leq 5$. Your plot should cover only that interval.

**(c)** Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^5 G_p(x) G_q(x)\, dx = 0, \ \ if\ p \neq q,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions that correspond to two distinctive eigenvalues $p$ and $q$? Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x)G_q(x)$ and show that it leads to a non-zero value even when $p \neq q$. One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all pairs of $p$ and $q$.

**Task 2** (3 points)
For $u(x,t)$ defined on the domain of $0 \leq x \leq 2\pi$ and $t \geq 0$, consider the PDE (in which $U$ and $K$ are constants)

$$\frac{\partial u}{\partial t} = U \frac{\partial u}{\partial x} + K \frac{\partial^2 u}{\partial x^2}$$

with periodic boundary conditions in the $x$-direction (i.e., $u(0, t) = u(2\pi, t)$, $u_x(0, t) = u_x(2\pi, t)$, and so on), and the boundary condition in the $t$-direction given as

$$u(x, 0) = \frac{[1-\cos(x)]^8}{256} \ .$$

Solve the PDE by Fourier series expansion. Plot the solution $u(x, t)$ at $t = 1$ for the three cases with (i) $U = -1$, $K = 0$, (ii) $U = 2$, $K = 0$, and (iii) $U = -1$, $K = 0.4$. Also, plot the solution at $t = 0$ (which is the same for all three cases). Please collect all four curves in one plot. [In the computation for making the plot, the Fourier series needs to be truncated. See the general remark below HW1-Task 1 on estimating the appropriate number of terms to keep in the series.]

**Task 3** (2.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$(1+t)\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^3 u}{\partial x^3} + 2\,u$$

with periodic boundary conditions in the $x$-direction, and the boundary condition in the $t$-direction given as

$$u(x, 0) = 1 + \cos(x) \ .$$

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals. The solution should be real and should be expressed all in real numbers and functions.

**Task 4** (2.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = 2\,\frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^8 u}{\partial x^8} - 4\,u$$

with periodic boundary conditions in the $x$-direction and the following boundary conditions in the $t$-direction,

(i) $u(x, 0) = 1 + 2\sin(x)$ ,

(ii) $u_t(x, 0) = 3 + 4\cos(x)$ .

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals. The solution should be real and should be expressed in real numbers and functions.