MAE/MSE502, Spring 2018 Homework #2

Hard copy of report is due 6:00 PM on the due date. Computer codes used to complete the tasks should be included in the report.

Task 0 (no point, but mandatory to complete for the report to be accepted)
Provide a statement to address whether collaboration occurred in completing this assignment. **This statement must be placed in the beginning of the first page of report.** See related clarifications in Homework #1.

Task 1 (3 points)
Consider the eigenvalue problem for $G(x)$ defined on the interval, $0 \le x \le 5$,

$$\frac{d^2G}{dx^2} = c\,G \ , \ G(0) = 0 \,, \, G'(5) = 3 \,.$$ (Note that the 2nd b.c. is imposed on the derivative of G.)

(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem. Are the eigenvalues discrete? For example, in the familiar problem when $G(x)$ is defined on $0 \le x \le 1$ and the boundary conditions are $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2\pi^2$ (n is a positive integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values within an interval, $A \le c \le B$, are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

(b) Plot the eigenfunctions, $G_C(x)$, associated to the eigenvalues c = –5, –1, –0.2, 0, 0.2, 1, and 5. (You will find in Part (a) that all those values are indeed valid eigenvalues.) Please collect all 7 curves in a single plot. Note that $G(x)$ is defined only on the interval of $0 \le x \le 5$. Your plot should cover only that interval.

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^5 G_p(x) G_q(x)\, dx = 0 \,, \ \ if \ \ p \ne q \ \ ,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions that correspond to two distinctive eigenvalues p and q? Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x)G_q(x)$ and show that it leads to a non-zero value even when $p \ne q$. One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all pairs of p and q.

(d) If $G_p(x)$ is an eigenfunction corresponding to an eigenvalue, $c = p$, would $AG_p(x)$ (where A is an arbitrary constant; $A \ne 1$) also be an eigenfunction? Provide a brief explanation to support your yes/no answer.

Task 2 (3 points)
For $u(x, t)$ defined on the square domain of $0 \le x \le 1$ and $0 \le t$, solve the PDE (which is modified from a wave equation),

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \pi^2 u$$

with the boundary conditions (u_x and u_t denote $\partial u/\partial x$ and $\partial u/\partial t$, respectively) ,

(i) $u_x(0, t) = 0$ (ii) $u_x(1, t) = 0$ (iii) $u(x, 0) = 3 + \cos(2\pi x)$ (iv) $u_t(x, 0) = 4 + \cos(\pi x)$

For this problem, we expect a closed-form analytic solution with a finite number of terms and with no unevaluated integrals. A deduction will be assessed on any unevaluated integral or a sum of infinitely many terms that is left untreated in the final answer.

Task 3 (3 points)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

with the boundary conditions (u_x and u_y denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively),

(i) $u_x(0, y) = 0$ (ii) $u_x(1, y) = \cos(2\pi y)$ (iii) $u_y(x, 0) = 0$ (iii) $u_y(x, 1) = 0$

(a) Test the solvability condition on the system. Based on it, which of the following is true?
(I) The system has no solution. (II) The system has a unique solution. (III) The system has multiple solutions. If your answer is (I), no need to proceed further. If your answer is (II) or (III), proceed to Part (b) and (c).
(b) Find the solution(s) of the system. If your answer for Part (a) is (III), please clearly write out what those multiple solutions are.
(c) Make a contour plot for one solution, $u(x,y)$, that you obtained from Part (b). If there are multiple solutions, just choose one to make the plot. See further instructions in the next page on using Matlab to make contour plots. The ideal spacing of contour levels is about 1/10 of the difference between the maximum and minimum values in the domain. Please adjust the contour levels such that the plot clearly demonstrates the structure of the solution, $u(x,y)$.

For Part (b) of Task 3, we expect a closed-form analytic solution with a finite number of terms and with no unevaluated integrals. A deduction will be assessed on any unevaluated integral or a sum of infinitely many terms that is left untreated in the final answer.

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x)\exp(-2y)$ over the domain of $0 \le x \le 1$ and $0 \le y \le 1$, using the contour levels of (-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9). The contours for u = -0.7, -0.3, 0.3, and 0.7 are labeled. It is essential to provide the coordinates of the grid (x2d and y2d in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y. A black-and-white contour plot is acceptable as long as the contours are properly labeled.

```
clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u,[-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h,[-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')
```