

## Another example for the method of characteristics

For  $u(x,t)$  defined on the infinite interval,  $-\infty < x < \infty$ , solve the PDE

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 ,$$

with the boundary condition,

$$u(x, 0) = P(x) \equiv \sin(x).$$

*Solution:*

Applying the method of characteristics, we have

$$dx/dt = u \quad (\text{the equation for the characteristics}) \quad (1)$$

$$du/dt = 0 \quad (u(t) \equiv u(x(t),t) \text{ is the solution along a characteristic } x(t) \text{ described by Eq. (1)}) \quad (2)$$

Solving (2), we have  $u(t) = u(0)$ , or  $u(x(t),t) = u(x(0),0) = P(x(0)) = \sin(x(0))$ . We will hereafter denote  $x(0)$  by  $x_0$  and  $x(t)$  by  $x$  so we have

$$u(x,t) = \sin(x_0) . \quad (3)$$

Solving (2) (and given that  $u(t) = u(0)$ ), we have  $x(t) = x(0) + u(0)t$ , or

$$x = x_0 + \sin(x_0) t \quad (4)$$

Equations (3) and (4) form the basis for us to evaluate the solution  $u(x,t)$  for a given  $(x,t)$ . This can be done in two steps:

- (i) With the given  $(x,t)$ , we use Eq. (4) to find  $x_0$ . Since Eq. (4) is a nonlinear equation in  $x_0$ , a numerical method (e.g., Newton's method, bisection method) can be used to find the solution. Note that the equation could have multiple solutions. We will discuss the issue shortly.
- (ii) Once  $x_0$  is obtained, the final solution is just  $u(x,t) = \sin(x_0)$  for the given  $(x,t)$ .

Since the solution is periodic in  $x$ , it suffices to find the solution within the interval of  $0 \leq x \leq 2\pi$ . Figure 1 shows the solutions at 0.3 and 0.7 obtained using the matlab code in the next page. Bisection method is used in the code to solve Eq. (4).

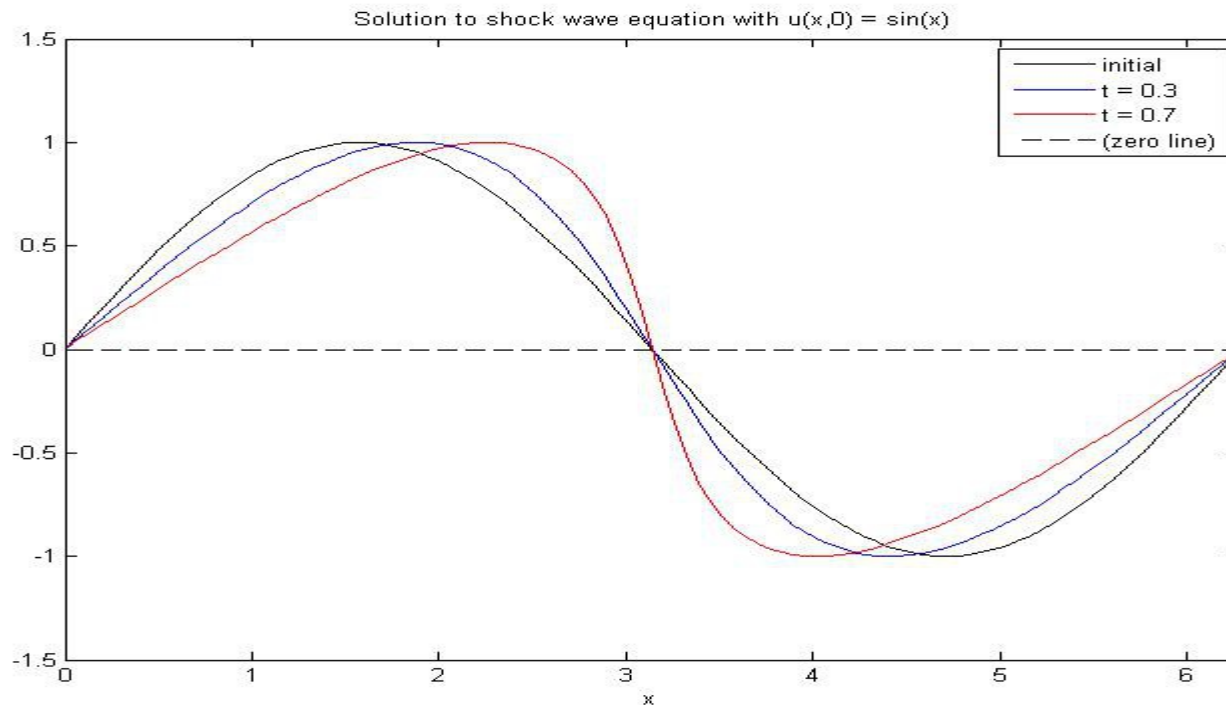


Fig. 1

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for k = 1:101
    x(k) = (k-1)*0.01*2*pi;
    u0(k) = sin(x(k));
    z(k) = 0;
end
t = 0.3;
for k = 1:101
    x(k) = (k-1)*0.01*2*pi;
    a = 1/t;
    b = x(k)/t;
    x1 = 0;
    xr = 2*pi;
    xm = (x1+xr)/2;
    for ip = 1:10
        if ((b-a*x1-sin(x1))*(b-a*xm-sin(xm)) <= 0)
            xr = xm;
        else
            x1 = xm;
        end
        xm = (x1+xr)/2;
    end
    x0 = xm;
    u1(k) = sin(x0);
end
t = 0.7;
for k = 1:101
    x(k) = (k-1)*0.01*2*pi;
    a = 1/t;
    b = x(k)/t;
    x1 = 0;
    xr = 2*pi;
    xm = (x1+xr)/2;
    for ip = 1:10
        if ((b-a*x1-sin(x1))*(b-a*xm-sin(xm)) <= 0)
            xr = xm;
        else
            x1 = xm;
        end
        xm = (x1+xr)/2;
    end
    x0 = xm;
    u2(k) = sin(x0);
end
plot(x,u0,'k-',x,u1,'b-',x,u2,'r-',x,z,'k--')
axis([0 2*pi -1.5 1.5])

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A special analysis for the behavior of the solution at  $x = \pi$  :

At  $x = \pi$ , Eq. (4) is reduced to

$$-x_0/t + \pi/t = \sin(x_0) \quad (5),$$

which always admits the solution of  $x_0 = \pi$ . Therefore,  $u(\pi, t) = u(\pi, 0) = 0$  is always a solution at  $x = \pi$ . (This behavior is reproduced in Fig. 1.) The question is whether it is the only solution. Note that for a given  $t$ , the solution  $x_0$  for Eq. (5) is the intersection of the line with a slope of  $-1/t$ ,  $-x_0/t + \pi/t$ , and the sinusoidal curve,  $\sin(x_0)$ . See sketch in Fig. 2. With a small  $t$ , we anticipate only one solution of  $x_0 = \pi$ . As  $t$  increases, specifically when  $t > 1$ , multiple solutions for  $x_0$  emerge. In physical space, this corresponds to multiple solutions for  $u(x, t)$  at  $x = \pi$ , as illustrated in Fig. 3. This situation can be physically meaningful. For example, if  $u(x, t)$  describes the elevation of the interface between two fluids, the interface is certainly allowed to "fold". In fact, it is for this type of problem that the method of characteristics becomes particularly useful. If other methods (e.g., Fourier method or just a general finite difference numerical method) are used to solve the problem, a finite-time blow up will occur as  $t$  approaches 1.

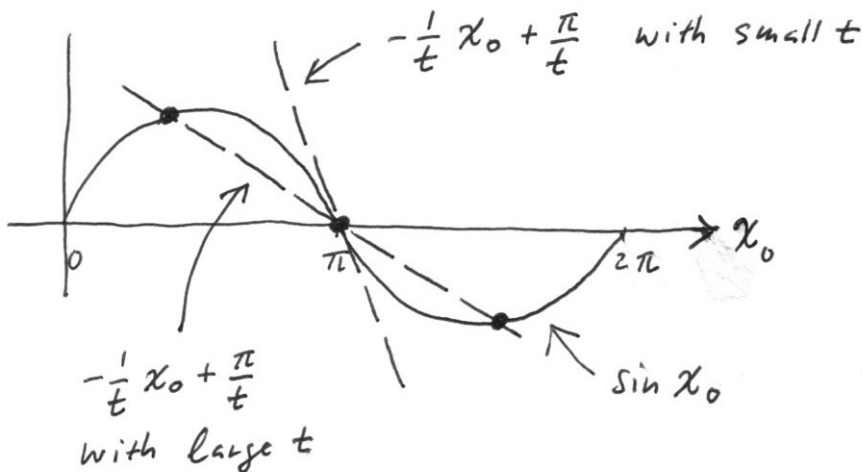


Fig. 2

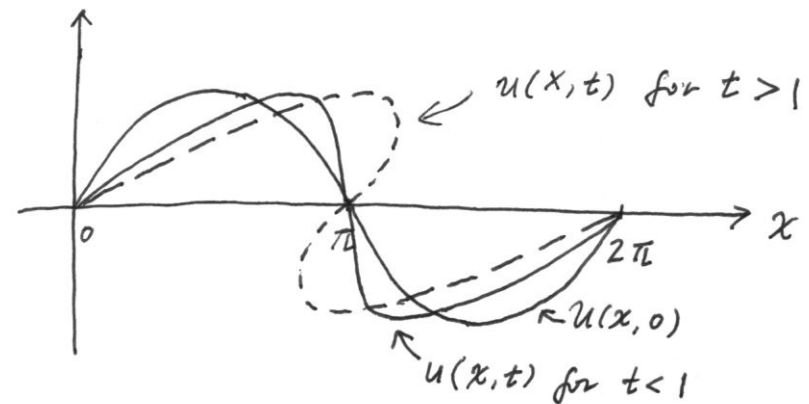


Fig. 3