This is a quick exercise equivalent to approximately 1/2 of a full homework assignment. This exercise is for refreshing your memory on the basic concept of numerical differentiation as a prerequisite of this course. Useful materials for this homework can be found in **Sec. 2.2** (from Eq. 2-1 to Eq. 2-16), **Sec 2.5** (Examples 2.2(a), 2.3, 2.4, 2.6, and 2.7), and **Sec. 2.6**. You might find **Sec. 2.6** particularly relevant.

For this problem, discussion with peers is allowed but the final write-up must be yours. Contribution from collaborator(s), if it is substantial, should be properly acknowledged.

1. (a) Using the formulas in the top two rows of Table 2.3 for the first and second derivatives, construct a <u>central</u> finite difference formula for the mixed partial derivative, $\partial^3 f/\partial x^2 \partial y$, with a truncation error of $O[(\Delta x)^2 (\Delta y)^2]$. Your final formula should read

$$\left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_{i,j} = \frac{A f_{i-1,j+1} + B f_{i-1,j-1} + C f_{i,j+1} + D f_{i,j-1} + E f_{i+1,j+1} + F f_{i+1,j-1}}{(\Delta x)^2 (\Delta y)} + O[(\Delta x)^2 , (\Delta y)^2] \ .$$

We use the notation in the textbook: $f_{i,j} \equiv f(i\Delta x, j\Delta y)$. Figure 1 illustrates the grid system. The open circle is where the mixed derivative will be evaluated. The triangles are where $f(x,y)$ needs to be evaluated in the finite difference formula. Please state clearly what your A, B, C, D, E, and F are according to the above formula.

(b) Similar to (a) but use the formulas in the top two rows of Table 2.1 to construct a <u>forward</u> finite difference formula for the mixed partial derivative, $\partial^3 f/\partial x^2 \partial y$, with a truncation error of $O[(\Delta x) (\Delta y)]$. Your formula should read

$$\left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_{i,j} = \frac{A f_{i+2,j+1} + B f_{i+1,j+1} + C f_{i,j+1} + D f_{i+2,j} + E f_{i+1,j} + F f_{i,j}}{(\Delta x)^2 (\Delta y)} + O[(\Delta x) , (\Delta y)] \ .$$

See Fig. 2 for an illustration of the grid system. Please state clearly what your A, B, C, D, E, and F are according to the above formula.

(c) Given $f(x,y) = \exp(y - x^2)$, use the 2nd-order and 1st-order formulas you obtained in (a) and (b) to evaluate $\partial^3 f/\partial x^2 \partial y$, at $x = 1$, $y = 1.5$. For simplicity, consider $\Delta x = \Delta y = \Delta s$ and perform the calculation for $\Delta s = 0.1$, 0.03, 0.01, 0.003, and 0.001. Evaluate the numerical error, $E = |$ numerical value − exact value $|$, associated with the five values of Δs for the two schemes. If you have derived the finite difference formulas correctly, for the 1st-order scheme E should vary linearly with Δs while for the 2nd order scheme E should vary with $(\Delta s)^2$. In other words, in the plot of $\log(\Delta s)$ vs. $\log(E)$ one should see a slope of approximately 1 for the 1st order scheme and approximately 2 for the 2nd order scheme. (The "log" here is natural log, although the choice of the base is not critical.) Make this log-log plot (collect all 10 data points -- 5 for each scheme -- in one plot) and perform a linear least squares fit of the data for each scheme to verify that the slopes for the two schemes are indeed close to their expected values. (In Matlab, the function "polyfit" is useful for performing the least squares fit of data.) Please submit your codes for Part (c).
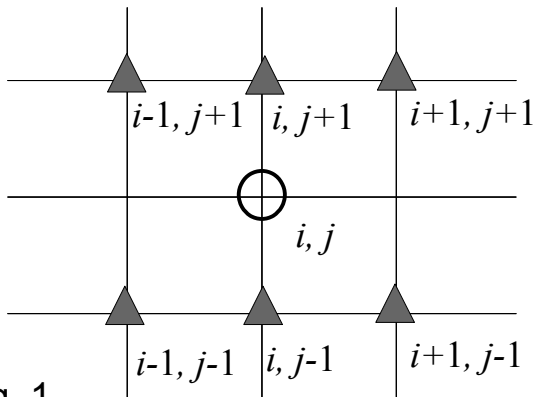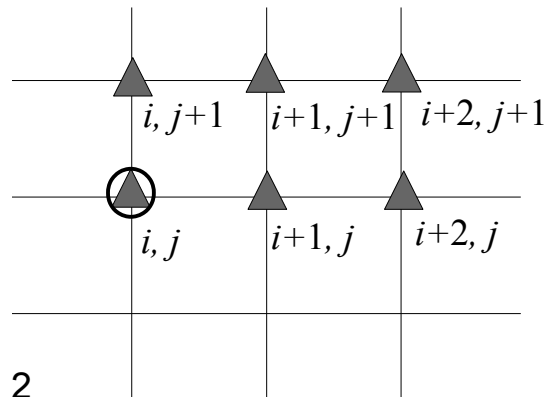

Fig. 1

Fig. 2