MAE561/471 Fall 2013 HW1

For this assignment, discussion with peers is allowed but the final write-up must be yours. Contribution from collaborator(s), if it is substantial, should be properly acknowledged.

Background:
For $u(x,t)$ defined on the domain of $0 \le x \le 20$ and $t \ge 0$, consider the linear advection equation with a constant C,

$$\frac{\partial u}{\partial t} = -C \frac{\partial u}{\partial x} \quad , \qquad (1)$$

and with the initial condition,

$$u(x,0) = \mathrm{P}(x),$$

where

$$\mathrm{P}(x) \equiv \cos[0.5\,\pi\,(x-5)]\ , \text{ if } 4 \le x \le 6$$
$$\equiv 0 \qquad , \text{ otherwise.}$$

This is the test case of the "advection of a bump" as we discussed in class. We will also assume periodic boundary conditions in x:

$$u(0, t) = u(20, t).$$

In this exercise, you will solve the above equation using four different finite difference schemes. Note that the analytic solution of the system is $u(x,t) = \mathrm{P}(x-Ct)$, which can be used to verify the numerical solutions. We consider $C = 1$ for all four problems. We will use i and j as the indices for x and t. The material for this exercise is briefly mentioned in **Sec 6.2** but we rely heavily on instructor's notes. All relevant transparencies are available online.

Please submit the print-out of your codes for all four problems. No code, no credit. As stated in our first-day lecture, computer programs in Matlab, Fortran, or C/C++ are acceptable.

1. *Forward upstream scheme.*
(a) Use the finite difference scheme,

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = -C\,\frac{u_{i,j} - u_{i-1,j}}{\Delta x} \quad ,$$

with $\Delta x = 0.05$ and $\Delta t = 0.01$, to obtain the solutions at $t = 4$, 8, 12, 16, and 20. Plot them as a function of x along with the initial state, $u(x, 0)$. Please collect all plots in one figure. Note that the solutions at $t = 4$, 8, 12 for this case have been discussed in class. (See Lecture 5 for reference plots.) At $t = 20$, the analytic solution is identical to the initial state. Therefore, the initial state can also serve as the analytic solution at $t = 20$ to verify the numerical solution.
(b) Choose $\Delta x = 0.005$ and $\Delta t = 0.001$ and obtain the solutions at $t = 40$, 80, 120, 160, and 200. Plot them along with the initial state, $u(x,0)$, all in one figure.
(c) Choose $\Delta x = 0.005$ and $\Delta t = 0.01$ and obtain the solutions at $t = 0.04$, and 0.08. Plot them along with the initial state. Comment on the behavior of the solution. (We will revisit this case in future lectures on numerical stability.)

(over please)

2. *Forward downstream* (*"FTFS" in the textbook*) *scheme.*
Use the finite difference scheme,

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = -C\,\frac{u_{i+1,j}-u_{i,j}}{\Delta x} \quad ,$$

with $\Delta x = 0.05$ and $\Delta t = 0.01$, to obtain the solutions at $t = 0.05$, 0.1, 0.15, and 0.2. Plot them along with the initial state, $u(x,0)$, all in one figure. (We have discussed in class that this is not a good scheme when $C > 0$. The solution at $t = 0.2$ is already quite corrupt and may contain very large values of u.)

3. *"Forward in time, central in space"* (*"FTCS" in the textbook*) *scheme*.
Use the finite difference scheme,

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = -C\,\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} \quad ,$$

with $\Delta x = 0.05$ and $\Delta t = 0.01$, to obtain the solutions at $t = 0.5$, 1, 1.5, and 2. Plot them along with the initial state. Comment on the behavior of the solution. (We will revisit this case in future lectures on numerical stability.)

4. *Lax-Wendroff scheme*.
Use the finite difference scheme,

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = -C\,\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} + \left(C^2\Delta t/2\right)\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2} \quad ,$$

with $\Delta x = 0.05$ and $\Delta t = 0.01$, to obtain the solutions at $t = 0.5$, 1, 1.5, and 2. Plot them along with the initial state. Compare the results with their counterparts in Prob. 3.