MAE561 Fall 2013 HW3

This is a full homework assignment for MAE561 participants and a half (1/2) assignment for MAE471 participants. MAE561 participants are required to solve both problems. MAE471 participants are required to solve only Prob 1 and will receive a bonus for a correct or nearly-correct solution of Prob 2. Bonus will be given only if the solution is of good quality. For Prob 1: **Please submit the print out of your codes. No code, no credit.** *For this assignment, discussion with peers is allowed but the final write-up must be yours. Contribution from collaborator(s), if it is substantial, should be properly acknowledged.*

1. In this problem we will solve Burgers' equation,

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} \quad ,$$

with the initial state given as

$$u(x,0) = \begin{cases} 2 \;, & \text{if } x \le 0 \\ 2 - x\;, & \text{if } 0 < x < 1 \\ 1 \;, & \text{if } x \ge 1 \;. \end{cases}$$

The analytic solution for the inviscid case ($\nu = 0$) is given in *Further information* in the next page. We will use the finite difference formula,

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = -\frac{E_{i,j} - E_{i-1,j}}{\Delta x} + \nu\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \quad ,$$

where $E \equiv u^2/2$. This is a combination of the FTBS scheme for the flux form of the inviscid Burgers equation, and a 2nd-order FTCS scheme for the viscous term. All notations are standard.
(a) Solve the inviscid ($\nu = 0$) version of the equation over the domain of $-1 \le x \le 3$, with the boundary conditions,

$$u(-1, t) = 2 \;, \quad u(3, t) = 1.$$

Using $\Delta x = 0.01$ and $\Delta t = 0.002$, obtain the numerical solutions at $t = 0.4$ and 0.8. Plot the numerical and analytic solutions at $t = 0.4$ and 0.8, along with the initial state (at $t = 0$), preferably in one figure. Repeat the calculation but with $\Delta x = 0.02$ and $\Delta t = 0.01$ and, again, plot the solutions at $t = 0.4$ and 0.8.
(b) Solve the full Burgers' equation over the domain, $-2 \le x \le 8$, with the boundary conditions,

$$u(-2, t) = 2 \;, \quad u(8, t) = 1 \;,$$

for the three cases with $\nu = 0$ (the inviscid equation), 0.05, and 0.2. Obtain and plot the solutions at t = 1.5 and 3 (please also plot the initial state) for all 3 cases. You do not need to plot the analytic solution for this part. In all cases, use $\Delta x = 0.01$. It is your job to choose appropriate values of Δt for the three cases to ensure numerical stability. State clearly what value of Δt you choose for each case.

2. In Prob 3 of HW1, we used the FTCS scheme to solve the linear advection equation defined on the interval of $0 \le x \le \mathrm{L}$. The finite-difference formula reads (all notations are standard)

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = -C\,\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \quad .$$

We also considered periodic boundary conditions,

$$u_{1,j} = u_{\mathrm{N},j} \;, \text{ for all } j \;,$$

where $i = 1$ and N correspond to $x = 0$ and L (therefore, $\Delta x = \mathrm{L}/(\mathrm{N}-1)$).

(a) Use von Neumann's method to show that the finite difference formula (along with the boundary conditions) described above is unconditionally unstable, i.e., it is always unstable regardless of the value of $\alpha \equiv C\Delta t/\Delta x$ used in the finite difference formula.

(b) Numerical instability implies that the "energy", defined as

$$E(j) = \sum_{i=1}^{N-1} (u_{i,j})^2 \quad ,$$

will keep increasing as j increases. Calculate $E(j)$ as a function of j for your numerical solution (with C = 1, Δx = 0.05, and Δt = 0.01) for Prob 3, HW1, to verify that $E(j)$ does increase with j. (Note that for the analytic solution $E(j)$ will remain constant for all time.) Perform two new runs for Prob 3, HW1, with Δt = 0.005 and Δt = 0.02 (C and Δx remain unchanged), then plot $E(j)$ as a function of j for all three cases with Δt = 0.005, 0.01, and 0.02, for the range of $0 \le t \le 2$. Does the artificial amplification due to numerical instability become more severe with an increasing α?

Further information

For $t < 1$, the analytic solution of the inviscid Burgers' equation with the initial condition given in Prob 1 can be obtained by the method of characteristics as

$$u(x, t) = \begin{cases} 2 & , \text{ if } x \le 2t \\ (2-x)/(1-t) & , \text{ if } 2t < x < 1+t \\ 1 & , \text{ if } x \ge 1+t \;. \end{cases}$$

See sketch below. This can be used to verify your numerical solution in Prob 1(a). As $t \to 1$, the initially sloping segment becomes vertical. For $t > 1$, the method of characteristics would produce multiple solutions with the "step" folding into a "Z" shaped structure (not shown). While multiple values of u at a given x can be meaningful for some special physical problems (e.g., if u represents the elevation of the interface of two different fluids), we expect a typical field variable (e.g., density, temperature, or velocity) of a fluid system to be single valued. In that case, once a discontinuity (or "front") of the variable is formed we need to impose an additional mathematical condition (called "shock condition") at the discontinuity to keep the solution physically meaningful. Without getting into the detail, this condition usually requires that conservation of energy ($u^2/2$ for our system) holds even at the discontinuity. Under that condition, the "front" will remain vertical but will propagate at a constant speed. In our problem, this speed (in the positive x direction) is Vs = 1.5. For example, at t = 2 the front will propagate to x = 3.5. See sketch. It is interesting to note that the numerical scheme used in Prob 1 actually emulates this solution for $t > 1$. (This is also the basis for several numerical tests, shown in Figs. 6.19-6.35, in Sec. 6.6 in the textbook.) Of course, in the numerical solution the "front" can never be perfectly vertical but would only reach a maximum (negative) slope of $-1/\Delta x$.