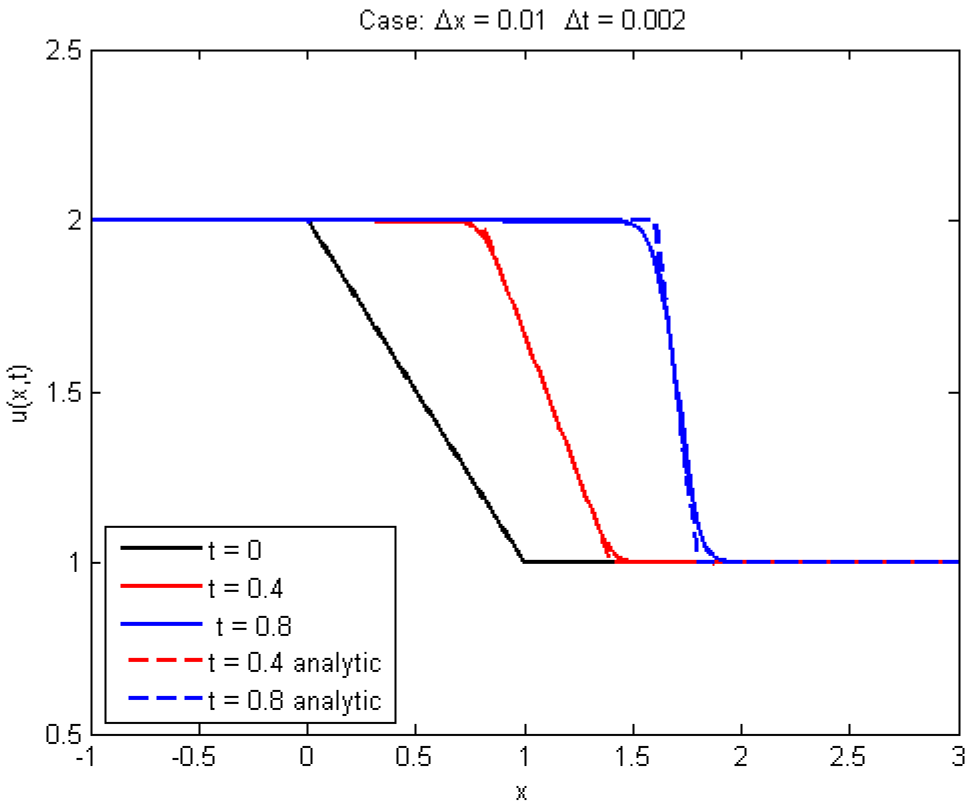
MAE561/471 HW3 solutions (prepared by HPH)

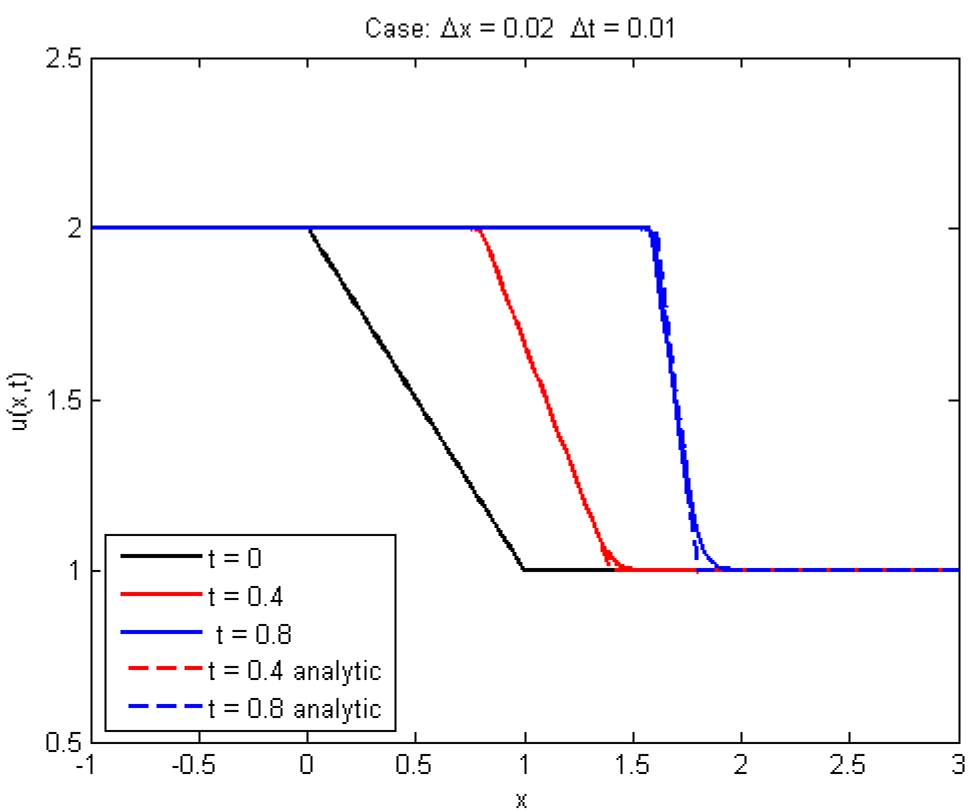
Prob 1a Matlab code for the first case ($\Delta x = 0.01$, $\Delta t = 0.002$)

```
clear;
dx = 0.01; dt = 0.002; A = dt/(2*dx);
x = [-1:dx:3]; N = length(x);
% -- initial condition
for i = 1:N
    if (x(i) <= 0)
        u(i) = 2;
    elseif (x(i) <= 1)
        u(i) = 2-x(i);
    else
        u(i) = 1;
    end
    uplot(1,i) = u(i);
end
% -- numerical integration in t
icount = 1;
for istep = 1:400
    for i = 3:N-1
        u_new(i) = u(i)-A*(u(i)^2)+A*(u(i-1)^2);
    end
    u_new(2) = u(2) - A*(u(2)^2)+A*(2^2);
    u_new(1) = 2;
    u_new(N) = 1;
    u = u_new;
    if (mod(istep,200) == 0)
      icount = icount+1;
      for i = 1:N
        uplot(icount,i) = u(i);
      end
    end
end
% -- analytic solution at t = 0.4 and 0.8
for it = 1:2
  icount = icount+1;
  t1 = it*0.4;
  for i = 1:N
    if (x(i) <= 2*t1)
        u(i) = 2;
    elseif (x(i) < 1+t1) && (x(i) > 2*t1)
        u(i) = (2-x(i))/(1-t1);
    else
        u(i) = 1;
    end
    uplot(icount,i) = u(i);
  end
end
plot(x,uplot(1,:),'k-',x,uplot(2,:),'r-',x,uplot(3,:),'b-',...
     x,uplot(4,:),'r--',x,uplot(5,:),'b--','LineWidth',2)
axis([-1 3 0.5 2.5])
xlabel('x');ylabel('u(x,t)')
title('Case: \Deltax = 0.01  \Deltat = 0.002')
legend('t = 0','t = 0.4',' t = 0.8', 't = 0.4 analytic',...
       't = 0.8 analytic','Location','SouthWest')
```

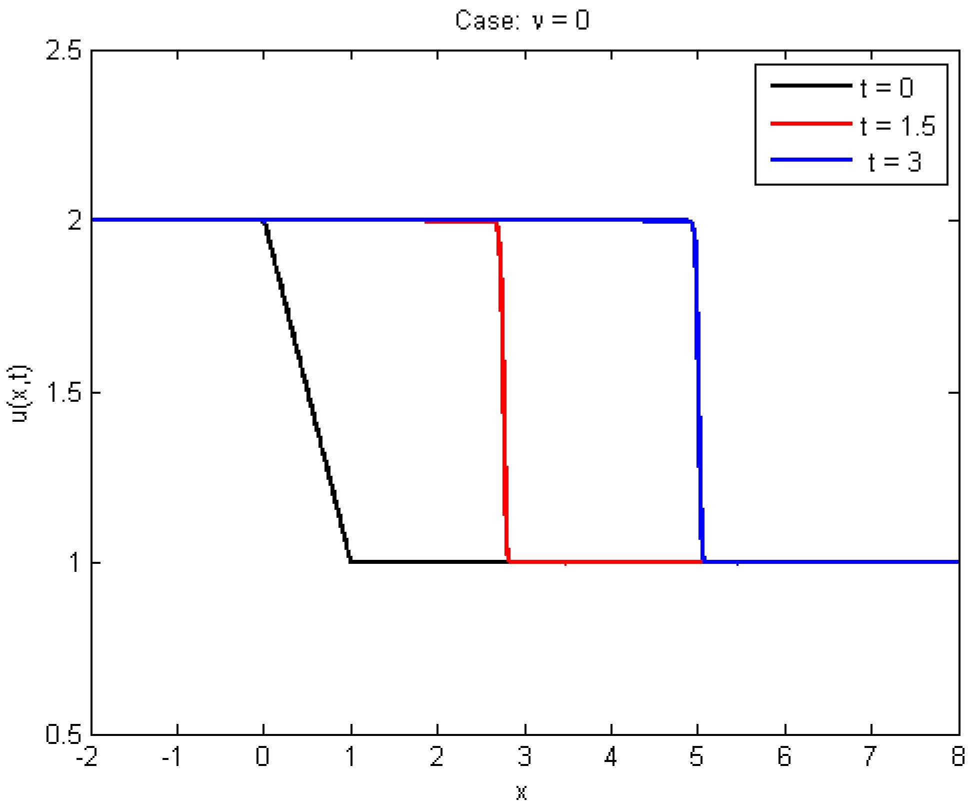
Prob 1a Plot for the case with ($\Delta x = 0.01$, $\Delta t = 0.002$)



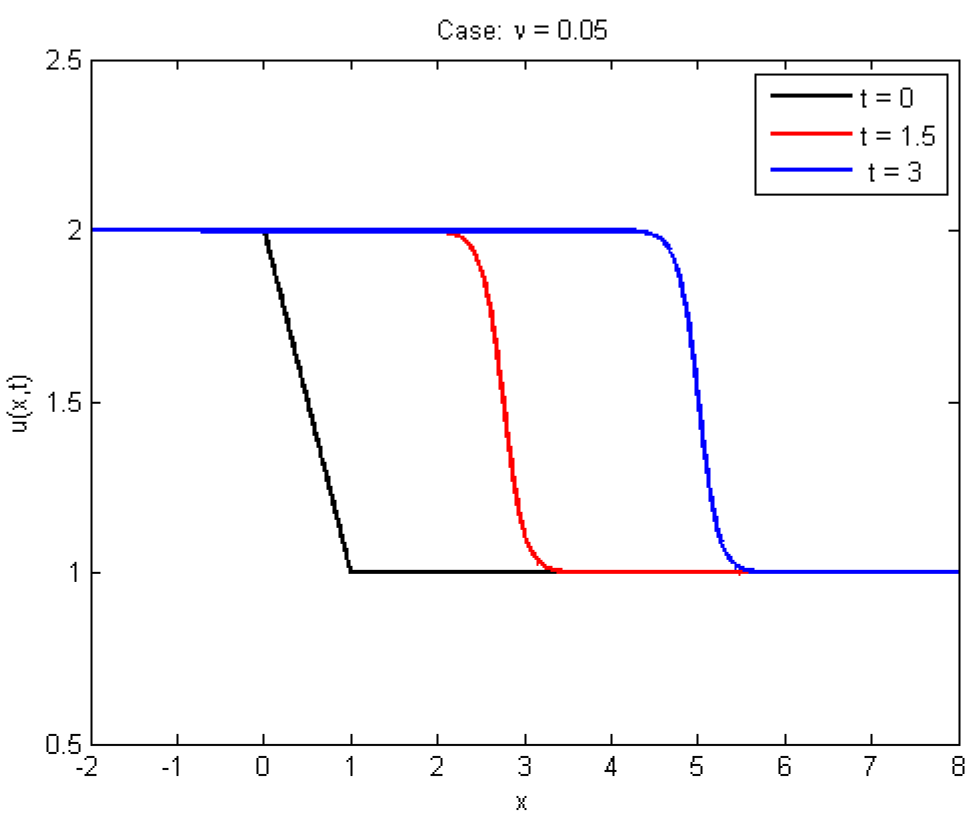
Prob 1a Plot for the case with ($\Delta x = 0.02$, $\Delta t = 0.01$)



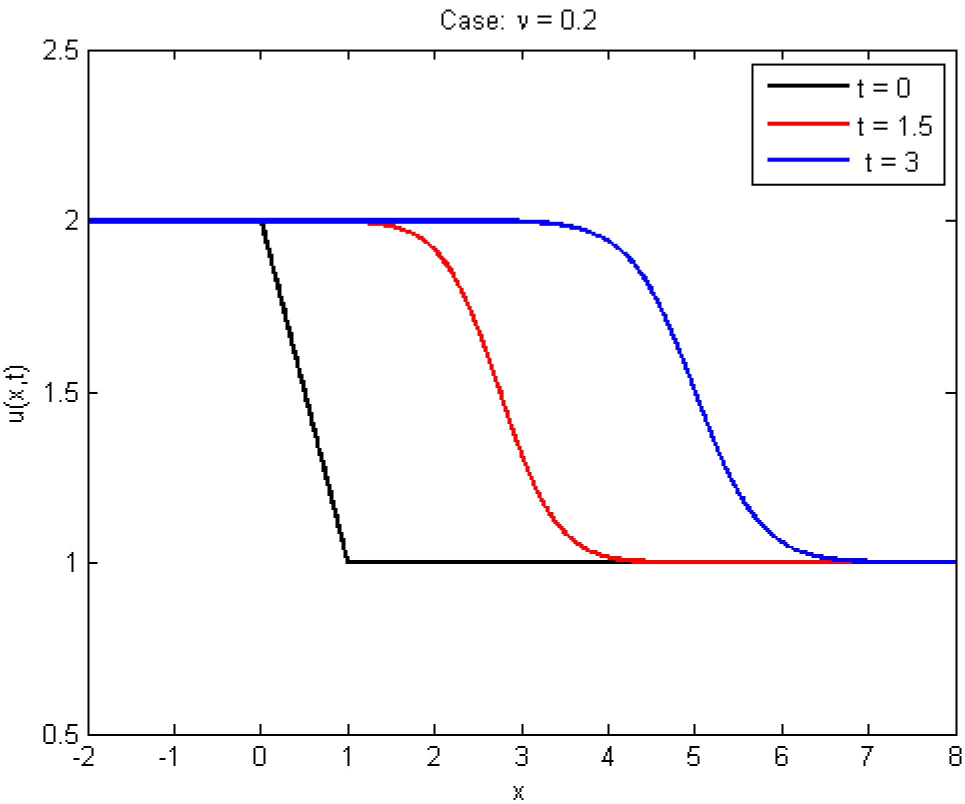
Prob 1b Plot for the case with ν = 0



Prob 1b Plot for the case with ν = 0.05



Prob 1b Plot for the case with $\nu = 0.2$



The solutions for Prob 1b are obtained using $\Delta t = 0.0001$. A smaller Δt should work for the cases with a smaller or zero viscosity.

Prob 2a

Plugging $u(x,t) \sim U_k(n)\, e^{ikj\Delta x}$ (j and n are the indices for x and t) into the finite difference formula, we obtain

$$\frac{U_k(n+1)\, e^{ikj\Delta x} - U_k(n)\, e^{ikj\Delta x}}{\Delta t} = -c\, \frac{U_k(n)\, e^{ik(j+1)\Delta x} - U_k(n)\, e^{ik(j-1)\Delta x}}{2\Delta x},$$

or,

$$\lambda_k \equiv \frac{U_k(n+1)}{U_k(n)} = 1 - \frac{\alpha}{2} e^{ik\Delta x} + \frac{\alpha}{2} e^{-ik\Delta x}$$

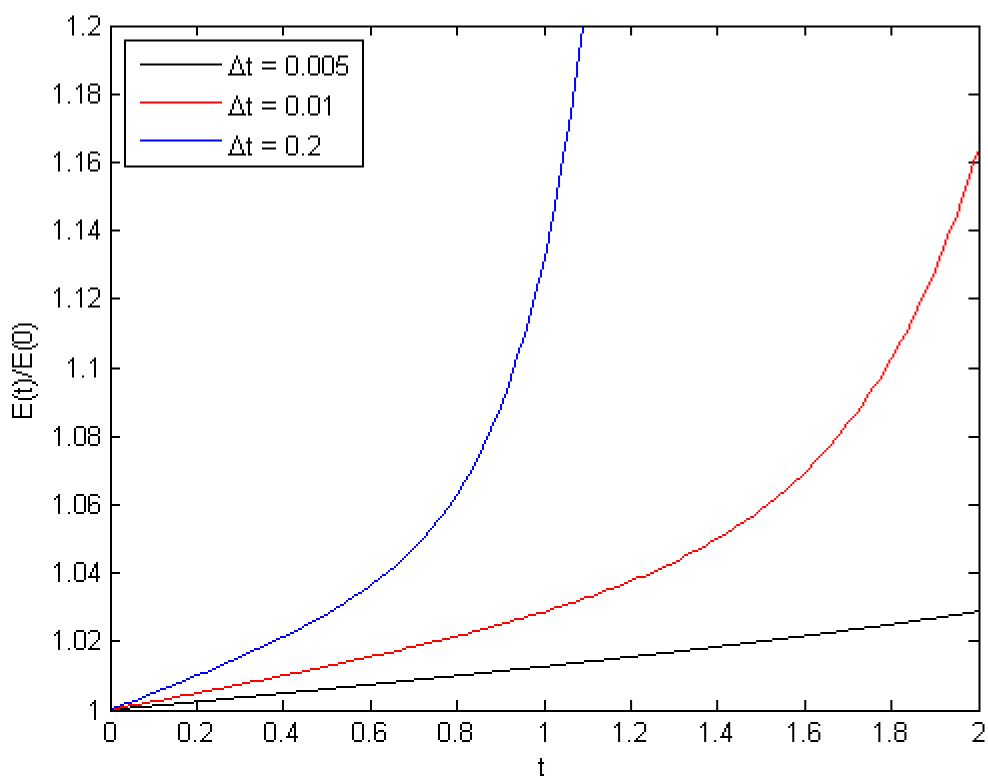
$$= 1 - i\alpha \sin(k\Delta x), \quad \text{where } \alpha \equiv \frac{c\Delta t}{\Delta x}$$

$$\Rightarrow \|\lambda_k\|^2 = 1 + \alpha^2 \sin^2(k\Delta x) > 1$$

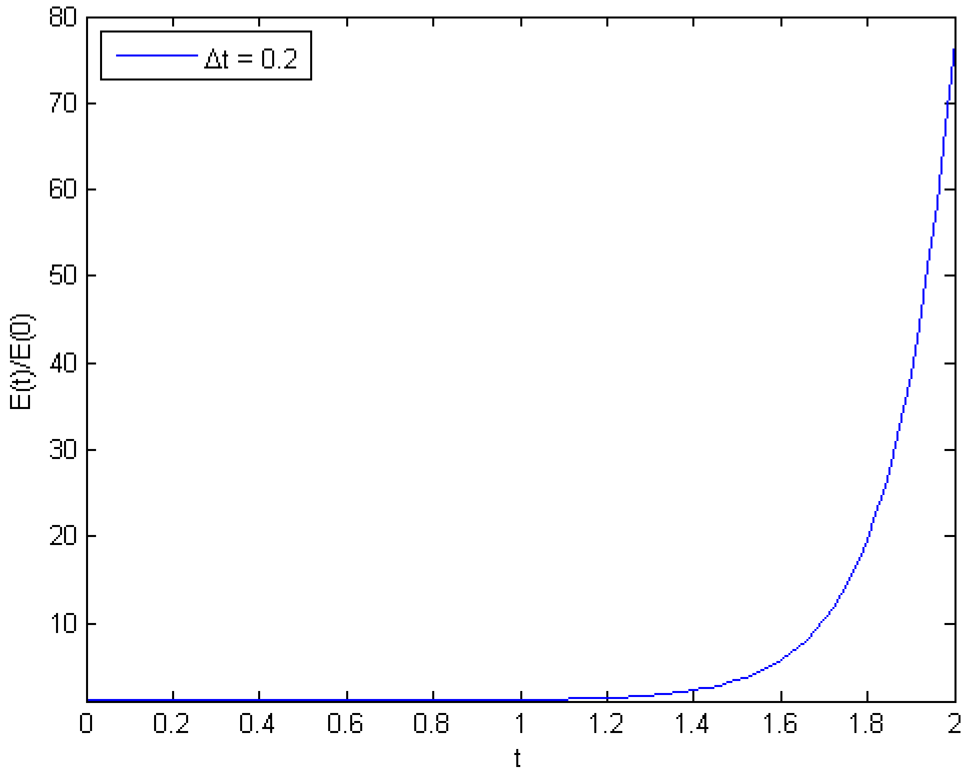
$\Rightarrow$ Unconditionally unstable.

Prob 2b

E(t)/E(0) as a function of t, for the case with Δt = 0.005, 0.01, and 0.02. (A separate plot for the case with Δt = 0.02 is in the next page.) Note that E(0) itself is around 20.



Prob 2b
E(t)/E(0) as a function of t, for the case with Δt = 0.02. The artificial amplification becomes more severe with an increasing Δt (or an increasing α with a fixed Δx).



Extra information (you didn't have to calculate this): log_{10}[E(t+Δt)/E(t) − 1] as a function of t for the three cases. We will discuss it in class.