

## MAE561 Fall 2013 HW4

For both problems: **Please submit the print out of your codes. No code, no credit.** For this assignment, discussion with peers is allowed but the final write-up must be yours. Contribution from collaborator(s), if it is substantial, should be properly acknowledged.

1. In this problem, we will solve the 2-D linear advection equation (with both  $C$  and  $D$  constants),

$$\frac{\partial u}{\partial t} = -C \frac{\partial u}{\partial x} - D \frac{\partial u}{\partial y} ,$$

defined on the square domain of  $0 \leq x \leq 5$  and  $0 \leq y \leq 5$ . The initial state is given as  $u(x, y, 0) = P(x, y)$ , where

$$P(x, y) = \begin{cases} \cos(0.5 \pi \delta) & , \text{ if } \delta \leq 1 \\ 0 & , \text{ otherwise } , \end{cases}$$

and

$$\delta \equiv \sqrt{(x-x_0)^2 + (y-y_0)^2} . \quad \text{Eq. (1)}$$

(The initial state is a 2-D circular bump centered at  $(x_0, y_0)$ .) We will use  $x_0 = 1.5$  and  $y_0 = 3$  through this homework. Note that the analytic solution of the equation is  $u(x, y, t) = P(x-Ct, y-Dt)$ . Periodic boundary conditions are imposed in both  $x$  and  $y$  directions,

$$u(5, y, t) = u(0, y, t) ,$$

$$u(x, 5, t) = u(x, 0, t) .$$

(a) Solve the case with  $C = 1$  and  $D = 0.5$  by using the 2-D version of the FTBS scheme,

$$\frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta t} = -C \frac{u_{i,j,k} - u_{i-1,j,k}}{\Delta x} - D \frac{u_{i,j,k} - u_{i,j-1,k}}{\Delta y} , \quad \text{Eq. (2)}$$

where  $(i, j, k)$  are the indices for  $(x, y, t)$ , with  $\Delta x = 0.02$ ,  $\Delta y = 0.02$ , and  $\Delta t = 0.01$ . Make contour plots of the solution  $u(x, y, t)$  at  $t = 0$  (the initial state), 2, 4, and 6. Please use  $(0.1, 0.2, 0.3, \dots, 0.8, 0.9)$  as the contour levels for all plots.

(b) **Bonus question:** Keep the setting in Part (a) but change  $D$  to  $-0.5$ . Try to integrate the system for a short time, make plots of the numerical solution  $u(x, y, t)$  at  $t = 0.2$  and  $0.4$ , and describe the behavior of the solution. You will find that the numerical scheme in Eq. (2) is no longer suitable for solving this case. Try to modify the numerical scheme to make it work. Write out your modified numerical scheme clearly and provide the reasoning for the modification. Based on the modified scheme, show your solution as contour plots of  $u(x, y, t)$  at  $t = 0, 2, 4$ , and  $6$ . Bonus will be given only if your modified scheme produces a solution of good quality (e.g., comparable to the quality of the solution in Part (a)) at  $t = 6$ .

2. In this problem, we will solve the 2-D diffusion equation,

$$\frac{\partial u}{\partial t} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) ,$$

defined on the square domain of  $0 \leq x \leq 5$  and  $0 \leq y \leq 5$ . The initial condition is identical to that given in Prob 1. The boundary conditions are given as

$$\begin{aligned}
u(0, y, t) &= 0 \\
u(5, y, t) &= 0 \\
u(x, 0, t) &= 0 \\
u(x, 5, t) &= 0.
\end{aligned}$$

Essentially,  $u$  vanishes at the boundary of the domain. Using the FTCS scheme,

$$\frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta t} = v \left( \frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta y)^2} \right) \quad (\text{typo corrected}),$$

with  $\Delta x = \Delta y = 0.05$  and  $\Delta t = 0.02$ , solve the case with  $v = 0.01$  and make contour plots of the solution  $u(x, y, t)$  at  $t = 0$  (the initial state), 8, 16, and 24. Please use (0.01, 0.1, 0.2, 0.3, ...) as the contour levels for all of your plots. See *Further information* below on how to impose specific contour levels in Matlab. The inclusion of a contour for  $u = 0.01$  is needed to clearly show the structure of the solution at large  $t$ .

### Further information

The following Matlab code makes a contour plot for  $u(x,y) = \sin(2\pi x)\exp(-2y)$  for the domain of  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , using the contour levels of (-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9). The contours for  $u = -0.7, -0.3, 0.3$ , and  $0.7$  are labeled.

```

clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u, [-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h, [-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')

```

