MAE561/471 Fall 2013 Homework #5

1. (a) For $\psi(x, y)$ defined on the domain of $0 \le x \le 5$ and $0 \le y \le 5$, solve the Poisson equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \Omega \quad , \qquad \text{Eq. (1)}$$

with Ω given as

$$\begin{aligned} \Omega(x, y) &= -\cos(0.5\,\pi\,\delta_1)\,,\ \text{if } \delta_1 \le 1 \quad \text{(beware of the minus sign in front of cos)} \\ &= 1.2\cos(\pi\,\delta_2)\,,\ \ \text{if } \delta_2 \le 0.5 \\ &= 0\,, \qquad \text{otherwise}\,, \end{aligned}$$

where

$$\delta_1 \equiv \sqrt{(x-1.5)^2+(y-3)^2} \quad \text{and} \quad \delta_2 \equiv \sqrt{(x-3)^2+(y-1.5)^2} \quad .$$

The boundary conditions are given as

$$\psi(0, y) = 0 \ , \ \ \psi(5, y) = 0\ , \ \ \psi(x, 0) = 0\ , \ \text{and} \ \ \psi(x, 5) = 0\ .$$

Essentially, ψ vanishes at the boundary of the square domain. Using a grid system with $\Delta x = \Delta y =$ 0.05, find the solution by the Point Gauss-Seidel iteration method (Sec 5.3.2 in the textbook) with the initial guess of $\psi(x, y) \equiv 0$. Make contour plots of the solution after 100, 200, 1000, and 2000 iterations. The recommended contour interval is 0.03 for ψ. Please also make a contour plot of $\Omega(x, y)$ as a reference.

(b) If ψ is the streamfunction of a 2-D incompressible flow, the 2-D velocity, $\mathbf{V} \equiv (u, v)$, is related to ψ by $u \equiv -\partial\psi/\partial y$ and $v \equiv \partial\psi/\partial x$. Use your solution of ψ after 2000 iterations in Part (a) to evaluate the corresponding (u, v) and make a plot of the vector field of $\mathbf{V}$. See *Further Information* in the next page for an example of using the "quiver" function in Matlab to plot a vector field.

2. Recognizing that the solution of the Poisson equation, Eq. (1), is the *stead state solution* (i.e., the solution as $t \to \infty$) of the following time-dependent PDE,

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \Omega \quad , \qquad \text{Eq. (2)}$$

we may alternatively obtain the solution of Eq. (1) by numerically integrating Eq. (2) to a large t. The finite difference formula for Eq. (2) can be constructed by setting the viscosity (ν) to 1 and adding a simple term for Ω in the right hand side of the formula given in HW4-Prob 2. Using the same domain, boundary conditions, grid system (with $\Delta x = \Delta y = 0.05$), and Ω as given in Prob 1, and the initial condition of $\psi(x, y, 0) \equiv 0$, find the numerical solution of Eq. (2) at $t = 0.5$, 1, 4, and 8. (It is your job to choose an appropriate Δt for the numerical integration.) Make contour plots of $\psi(x, y, t)$ at those four times using the same contour levels you used in Prob 1. Does your solution for Prob 2 agree with the solution for Prob 1?

Further Information: The following code demonstrates the use of the "quiver" function in Matlab. It plots the vector field of $\mathbf{V} = (u,v)$, where $u(x,y) = -2\,y$ and $v(x,y) = 2\,x$.

```
clear
x = [-2:0.2:2]; y = [-2:0.2:2];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = -2*y(j);
        v(i,j) = 2*x(i);
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
quiver(x2d,y2d,u,v)
axis([-2 2 -2 2])
xlabel('x');ylabel('y')
```