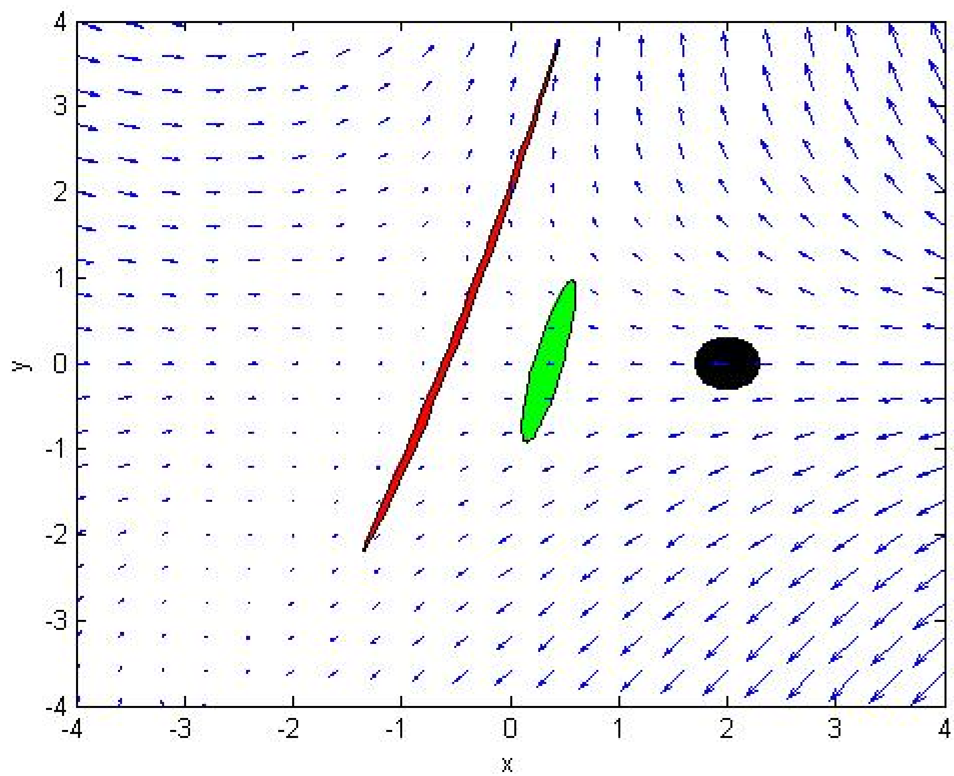
MAE571, Fall 2014, HW1 Solutions

Prob 1

The solution can be obtained by numerically integrating (dx/dt, dy/dt) = (u, v), where (u, v) are already given. This has to be done repeatedly for a collection of initial points that circle the blob at t = 0. The positions of those initial points can be defined by $x_n(0) = 2 + 0.3\cos(\varphi_n)$, $y_n(0) = 0.3\sin(\varphi_n)$, and $\varphi_n = n\,\Delta\varphi$, n = 1, 2, 3, ... (this is just one of many possibilities). In the example of Matlab code (next page), we use 80 points to define the initial circle and use the Euler explicit scheme (with d/dt replaced by the first-order forward finite difference formula) with $\Delta t = 0.0001$ to perform the numerical integration for each point.

Plot:
(Black, green, and red are t = 0, 0.8, and 2.0)



Example of Matlab code for Prob 1

```
clear
% --- plot the vector field ---
x1d = [-4:0.4:4]; y1d = [-4:0.4:4];
for i = 1:length(x1d)
    for j = 1:length(y1d)
        x2d(i,j) = x1d(i);
        y2d(i,j) = y1d(j);
        u2d(i,j) = -x2d(i,j)+0.6*y2d(i,j)-1;
        v2d(i,j) = y2d(i,j)+0.4*x2d(i,j)*y2d(i,j);
    end
end
hold on
quiver(x2d,y2d,u2d,v2d,1)
axis([-4 4 -4 4])
% --- define the points that circle the dyed blob at t = 0 ---
N = 80;
dphi = 2*pi/N; r1 = 0.3;
for n = 1:N
    phin = (n-1)*dphi;
    x0(n) = 2+r1*cos(phin);
    y0(n) = r1*sin(phin);
end
% --- integration in time to t = 0.8 ---
for n = 1:N
    xt(n) = x0(n);
    yt(n) = y0(n);
end
t = 0.8; dt = 0.0001; Nt = t/dt;
for it = 1:Nt
    for n = 1:N
       xplus = xt(n)+(-xt(n)+0.6*yt(n)-1)*dt;
       yplus = yt(n)+( yt(n)+0.4*xt(n)*yt(n))*dt;
       xt(n) = xplus;
       yt(n) = yplus;
    end
end
% --- continue integration in time to t = 2 ---
for n = 1:N
    xt2(n) = xt(n);
    yt2(n) = yt(n);
end
t = 1.2; dt = 0.0001; Nt = t/dt;
for it = 1:Nt
    for n = 1:N
       xplus = xt2(n)+(-xt2(n)+0.6*yt2(n)-1)*dt;
       yplus = yt2(n)+( yt2(n)+0.4*xt2(n)*yt2(n))*dt;
       xt2(n) = xplus;
       yt2(n) = yplus;
    end
end
% --- fill the dyed blob at t = 0, 0.8, and 2.0 ---
fill(x0,y0, [0 0 0]);
fill(xt,yt, [0 1 0]);
fill(xt2,yt2,[1 0 0]);
xlabel('x'); ylabel('y')
box on
```

Prob 2

The rate of change of the volume of a Lagrangian parcel is related to the divergence of the flow field by $d(\ln V)/dt = \nabla\cdot\vec{v}$, where V is volume and $\vec{v}$ is the velocity vector. (See the slides for Lecture #3.) For a 2-D flow, the relation is reduced to $d(\ln A)/dt = (\nabla\cdot\vec{v})_{2D} \equiv \partial u/\partial x + \partial v/\partial y$, where A is the area. Note that this relation is purely kinematic and is true regardless of the property of the flow. For Prob 1, from the given velocity, (u, v), we have $\partial u/\partial x + \partial v/\partial y = 0.4\,x$. Therefore, $d(\ln A)/dt = 0.4\,x$ which is positive over the entire right half plane with a positive x. The area of the blob is not conserved. We expect the area of the blob to expand as it travels through the right half of the x-y plane. For the unsteady flow given in Prob 2, $\partial u/\partial x + \partial v/\partial y = 0$ at all time and all locations. The area of the blob is conserved. (Both conclusions can be corroborated by direct numerical evaluations of the area of the blob at different times.)

Prob 3

With the conditions given in the problem, at t = 0 the only term in the right hand side that is non-zero is the y-component of the solenoid term. Therefore, the initial tendency for ω_y can be readily derived as

$$\left(\frac{\partial\omega_y}{\partial t}\right)_{t=0} = -\,\boldsymbol{j}\cdot\left(\nabla\left(\frac{1}{\rho}\right)\times\nabla p\right) \; .$$

(Note that for t > 0 the above expression would not be true. More terms will contribute to the tendency.) Since $\nabla(1/\rho) = -\rho^{-2}\nabla\rho$ and $\nabla\rho\times\nabla p$ points "out of the paper" (i.e., in the negative y-direction) in Fig. 1, we have

$$\left(\frac{\partial\omega_y}{\partial t}\right)_{t=0} = -\boldsymbol{j}\bullet(-\boldsymbol{j})\,(-\rho^{-2}\;|\nabla\rho|\,|\nabla p|\;\sin(\theta)\,)\,,$$

where θ is the acute angle between the $\nabla\rho$ and ∇p vectors. From Fig. 1, we have $|\nabla\rho| = 0.01/100$, $|\nabla p| = 1000/100$, ρ = 1 (all in SI units) at the location marked by a star, and θ = 2°, which give the initial tendency as

$$\left(\frac{\partial\omega_y}{\partial t}\right)_{t=0} = -\,3.49 \times 10^{-5}\ \mathrm{s}^{-2}\,.$$

Prob 4

The report provided by the 1st team indicates that $\partial T/\partial x = -\,0.1$ °C/km. The mobile unit, which moved at a speed of u = 5 m/s, observed $dT/dt = -\,1$ °C/hr. Combining these pieces of information, the local rate of change of T should be $\partial T/\partial t = dT/dt - u\,\partial T/\partial x = 0.8$ °C/hr. Note that temperature actually increased with time at a fixed location.