MAE571, Fall 2014 Homework #2
Due Wednesday, October 22, at start of class

1. This exercise will use a simple 1-D model to illustrate that (not counting the effect of the boundaries) the viscous term only acts to redistribute momentum but does not destroy the total momentum of the fluid system. At the same time, the viscous term does dissipate kinetic energy (turning it into heat). Consider the system of a 1-D flow, $u(z,t)$, defined on the unbounded domain of $-\infty < z < \infty$,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} \quad ,$$

with the natural boundary conditions of $u \to 0$ and $\partial u/\partial z \to 0$ as $z \to \pm\infty$, and a given initial profile, $u(z,0)$. The system isolates the effect of viscosity on a unidirectional flow, $u(z,t)$, which does not vary in the $x$ and $y$ direction. The total momentum, $M$, and total kinetic energy, $E$, of the system are defined as

$$M(t) \equiv \int_{-\infty}^{\infty} u \, dz \quad , \quad E(t) \equiv \int_{-\infty}^{\infty} \left(\frac{1}{2}u^2\right) dz \quad .$$

(a) Show that, for the given boundary conditions, $M$ is conserved. (In other words, $dM/dt = 0$.) Show that the evolution of $E(t)$ can be expressed as

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \nu \, u \, \frac{\partial^2 u}{\partial z^2} \, dz \quad . \qquad \text{Eq. (1)}$$

Equation (1) is an example of the "viscous dissipation of energy" discussed in Sec 6.5 in the textbook. (20%)

(b) Setting $\nu = 1$ and giving the initial profile of $u$ as $u(z,0) = \exp(-z^2)$ (a "Gaussian jet" as illustrated in Fig 1), find the solution, $u(z,t)$. Show that the solution indeed satisfies $dM/dt = 0$. Using the solution, evaluate $E(t)$ and make a plot of $E(t)$ vs. $t$. Show that asymptotically $E(t) \sim t^{-1/2}$ at large $t$. (For example, the value of $E(t)$ at $t = 400$ is approximately a half of that at $t = 100$. Thus, this is not a very effective process for dissipating kinetic energy.) (30%)

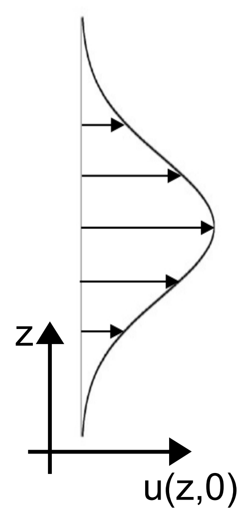
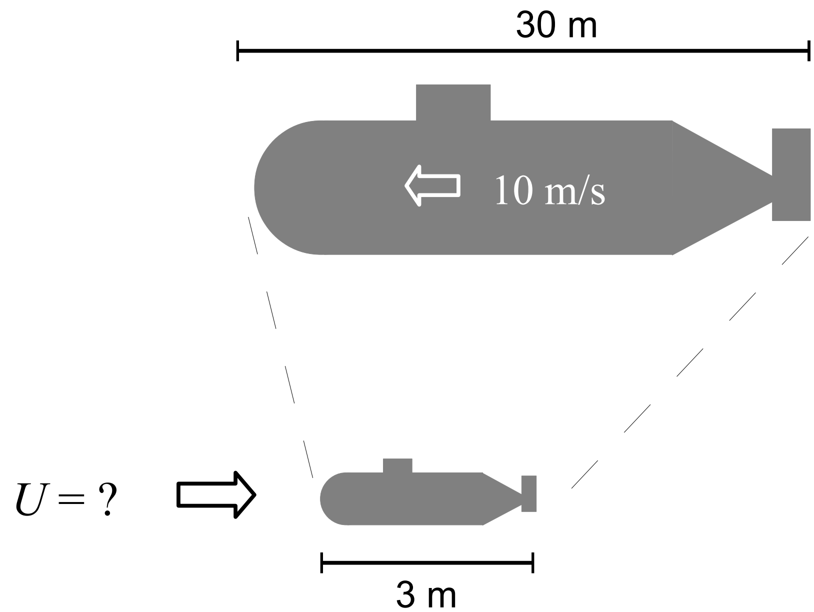



Fig. 1

2. A submarine is 30 m long and typically operates at a speed of 10 m/s undersea. If one attempts to test a 10:1 scaled-down model of the submarine (i.e., the toy model is 3 m long, with all other dimensions adjusted in proportion) in a wind tunnel, what would be the appropriate wind speed to impose in the tunnel in order for the flow around the toy submarine to emulate the flow around the real submarine moving at its typical speed? Here, we assume that both air and sea water are incompressible, that we are only interested in testing a steady flow or flow in statistical equilibrium, etc., such that "Reynolds number similarity" holds. (20%)




3. If $\vec{V}$ is the 3-D velocity and $G$ is any variable (e.g., density) of a fluid flow, show that, for any given volume $V$ that is fixed in space (i.e., $V$ is independent of time),

$$\frac{d}{dt}\iiint_V G\,dV = \iiint_V \left(\frac{dG}{dt} + G\nabla\cdot\vec{V}\right)dV - \oiint_S (\vec{V}G)\cdot\hat{\boldsymbol{n}}\,dS \quad , \qquad \text{Eq. (2)}$$

where all notations are conventional: d/d$t$ is the total derivative (not to be confused with partial derivative, $\partial/\partial t$) with respect to time, $S$ is the boundary of $V$, and $\hat{\boldsymbol{n}}$ is the outward normal vector at $S$. Note that the relation in Eq. (2) is purely kinematic. It does not depend on what kind of flow we have. (30%)

Remark: If the $V$ in Prob 3 is a control volume, i.e., if nothing moves across its boundary, then the 2nd term in the right hand side of Eq. (2) vanishes. In that case, we recover the rather trivial spacial case of Reynolds Transport Theorem (RTT) in which the control volume is fixed in space (i.e., $V$ is not a function of $t$). The derivation of the general RTT (p. 206 in the textbook) is more complicated since it allows the control volume to move. (Exercise 6.13 in the textbook provides some hints on the key step of the proof of the RTT.) A derivation of the RTT is not required for Prob 3. In fact, it should be reiterated that the $V$ given in Prob 3 is *any* volume (not necessarily a control volume) that is fixed in space. In the RTT as stated in the textbook, the $V(t)$ is a control volume that is allowed to move.