MAE571, Fall 2014 HW2 Solutions

Prob 1. (a)

$$\frac{dM}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} u\ dz = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\ dz = \int_{-\infty}^{\infty} \nu \frac{\partial^2 u}{\partial z^2}\ dz = \nu[\frac{\partial u}{\partial z}(\infty) - \frac{\partial u}{\partial z}(-\infty)] = 0.$$

$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}u^2\ dz = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\frac{1}{2}u^2)\ dz = \int_{-\infty}^{\infty} u\ \frac{\partial u}{\partial t} dz = \int_{-\infty}^{\infty} \nu\ u\ \frac{\partial^2 u}{\partial z^2}\ dz \quad .$$

(b) The solution is

$$u(z,t) = \frac{\exp(-z^2/(4t+1))}{\sqrt{4t+1}} \quad .$$

Using the solution, we can readily obtain $M(t) = \pi^{1/2}$, which is independent of t , and

$$E(t) = (\frac{\pi}{32t+8})^{1/2} \quad .$$

So, at large t, $E(t) \sim (\pi/32)^{1/2}\ t^{-1/2}$.

2. Forcing the Reynolds number of the system of the real submarine in the sea to equal its counterpart of the toy submarine in the wind tunnel, we find $U \sim$ 1400-1800 m/s depending on temperature and if regular water is considered. If salty water is considered, the value of U is somewhat smaller depending on salinity (Remark: This exercise illustrates the limitation of Reynolds number similarity. It actually would not work in this case since $U \sim$ 1800 m/s exceeds 5 times the speed of sound, which would render the system compressible. Thus, Reynolds number would no longer be the only controlling factor for the flow. For example, if we consider the Mach number, $Ma = U/c$ where c is the speed of sound, it will be over 5 for the toy submarine system but less than 0.01 for the real submarine in water. Clearly, the two systems are no longer dynamically equivalent to each other. (The speed of sound is approximately 340 m/s in air and 1500 m/s in regular water at 20 °C.)

3. Detailed derivation:

$$\frac{d}{dt}\iiint_V G\,dV = \iiint_V \frac{\partial G}{\partial t}\,dV \qquad \text{(basic calculus)}$$

$$= \iiint_V (\frac{dG}{dt} - \vec{V}\cdot\nabla G)\,dV \qquad \text{(Eulerian to Lagrangian)}$$

$$= \iiint_V (\frac{dG}{dt} + G\nabla\cdot\vec{V} - \nabla\cdot(G\vec{V}))\,dV \quad (\text{since } \nabla\cdot(G\vec{V}) \equiv G\nabla\cdot\vec{V} + \vec{V}\cdot\nabla G\ )$$

$$= \iiint_V (\frac{dG}{dt} + G\nabla\cdot\vec{V})\,dV - \oiint_S (\vec{V}G)\cdot\hat{\boldsymbol{n}}\,dS \quad \text{(by Gauss Divergence theorem)} .$$