MAE571, Fall 2016   Homework #1 (14 points)

Note: 1 point ~ 1% of the total score for the semester. Hard copy of report is due at the start of class on the due date. Please submit the printout of computer codes (in Matlab, Fortran ,C++, etc.) used in the work. Additional rules on collaboration will be released separately. Please always follow the rules.

1. A steady two-dimensional flow is given as

$$u(x, y) = -x - 0.2\,x^2 + 0.6\ \ y - 0.3$$
$$v(x, y) = \quad y + 0.4\,x\,y\,.$$

A blob of fluid initially (at $t = 0$) fills a circle centered at $(x, y) = (2, 0)$ with a radius of 0.3 (see Fig. 4 in **Additional Note**). Given the flow field, track the blob to $t = 0.6$ and 1.5 and make a plot to show the blob at $t = 0$, 0.6, and 1.5 in the fashion of Fig. 4. Please also superimpose the vectors of the (Eulerian) velocity field to the plot. Hint: It suffices to track the movement of the boundary points of the blob, then connect them to outline the blob at a given time. In Matlab, the "fill" command fills an area enclosed by a set of points. The vector field can be plotted by the "quiver" command in Matlab. See Additional Note for an example of Matlab code. (2 points)

2. Consider the same flow field as in Prob 1. Instead of a circular blob, place a "checkerboard" with 9 square elements at the initial time, as illustrated in Fig. 1. Specifically, the checkerboard is initially centered at $(x, y) = (2, 0)$ (indicated in Fig 1 as the red dot) with each square having the dimension of 0.5 x 0.5. Track the evolution of the checkerboard to $t = 1.0$ and make a plot of the checkerboard at $t = 0$ and 1.0, in the fashion of Fig. 1. (3 points)

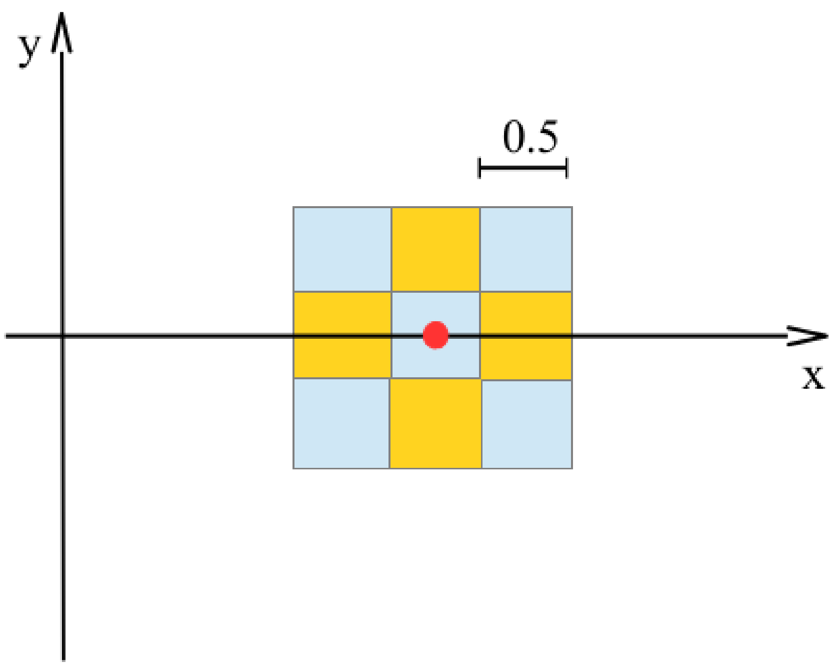


Fig. 1

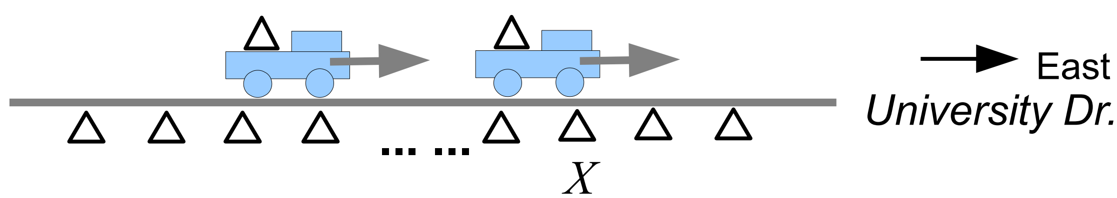
3. (a) In Prob. 1, will the "dyed blob" of fluid conserve its volume (or "area" since it is 2-D) as it moves and undergoes deformation? Must provide a rigorous reasoning to receive credit. A mere "yes" or "no" answer will receive zero point, even if it is correct.
(b) If the steady flow in Prob. 1 is replaced by an unsteady flow given as

$$u(x, y, t) = -x\,t + 0.6\,y - 3$$
$$v(x, y, t) = 3\,y\,t - 0.5\,x + 1\ ,$$

will the dyed blob conserve its area? If your answer is "no" for (b), compute the area of the blob (whose initial position and extent are given in Prob 1) at $t = 0.5$. (3 points)

4. A field campaign was conducted to measure the temperature across the city of Mesa. Two teams were deployed to measure the air temperature along University Drive which runs in the east-west direction through the city. The 1st team consisted of local volunteers whose houses happen to be located on University Dr. Thermometers (illustrated in Fig. 2 as the triangles) were set up in their front yards to monitor local temperature. The 2nd team operated a fleet of mobile units, each carrying a thermometer on board and all moving eastward along University Dr. at a constant speed of 10 m/s. The 1st team reported that at any given time during the campaign temperature <u>decreases eastward</u> along University Dr. at a constant rate of 0.2 °C/km. The 2nd team reported a constant <u>decrease</u> in temperature at a rate of 0.5 °C/hour as recorded by any of the mobile thermometers. If at the start of the campaign the temperature at a fixed location (for example, as marked by "*X*" in Fig. 2) on University Dr. was 25 °C, what would be the temperature at the same location one hour into the campaign? (1 point)

Fig. 2



5. *Background*: A sinkhole exists at the bottom of a lake such that water continuously "leaks" into it. The water level of the lake is maintained by an external supply of water (by precipitation, drainage into the lake, etc.) that balances the mass loss into the sinkhole. We assume that the floor of the lake is flat and ignore the movement of the free surface of the lake. The density of water is assumed to be constant. The effect of viscosity is ignored. In the vertical direction, the system is in hydrostatic balance with zero vertical velocity. Given the constant depth in the horizontal direction, the horizontal components of pressure gradient force vanish. Considering all of the above conditions, the flow in the lake could be described approximately by an idealized 2-D system (assuming that horizontal velocity is independent of depth) with its continuity and momentum equations given as

$$\nabla \cdot \mathbf{v} = -S\ , \qquad \text{Eq. (1)}$$

and

$$\partial \mathbf{v}/\partial t = -\,\mathbf{v} \cdot \nabla \mathbf{v}\ , \qquad \text{Eq. (2)}$$

where $S > 0$ is a constant, $\mathbf{v} \equiv (u, v)$ is the 2-D horizontal velocity vector and the "del operator", ∇, is also purely 2-D as defined by $\nabla \equiv \mathbf{i}\ \partial/\partial x + \mathbf{j}\ \partial/\partial y$. Equation (1) represents the idealization that the mass sink associated with the sinkhole drives a converging motion with a constant *negative divergence* of 2-D velocity. The geometry of the flow system is shown in Panel A of Fig. 3. Accepting Eqs. (1) and (2) as the starting point, answer the following questions. (5 points for (a)-(d) below)

(a) Given Eqs. (1) and (2), show that when $S = 0$ (i.e., when the sinkhole is closed) the flow system satisfies Lagrangian conservation of vorticity,

$$d\,\omega / d\,t = 0\,,$$

where the (2-D) vorticity is defined by $\omega \equiv \partial v/\partial x - \partial u/\partial y$. (See Section 1.4-1.5 in textbook.) In other words, vorticity (despite itself being a dynamic variable) behaves like a passive tracer.

(b) When $S \neq 0$ as described in *Background*, for any given Lagrangian fluid parcel with an initial vorticity of $\omega(0)$, find the exact expression of $\omega(t)$ in terms of $\omega(0)$ and S. Or, find the expression of the ratio, $\omega(t)/\omega(0)$, in terms of S. Does this ratio depend on whether the initial vorticity of the parcel, $\omega(0)$, is positive or negative? (In other words, would flipping the sign of $\omega(0)$ change the ratio at a given time?)

(c) Consider the situation, illustrated in Panel B of Fig. 3, when a thin vortex ring centered at the sinkhole is created at $t = 0$. The azimuthal velocity along the ring is initially axially symmetric. This ensures that the system is axially symmetric for all time. Let Ω be the total vorticity of the ring (which would shrink over time), i.e., Ω is the integral of ω over the ring. Since the system is axially symmetric, the behavior of $\Omega(t)/\Omega(0)$ will be the same as $\omega(t)/\omega(0)$ as determined from (b). [One can reach this conclusion by considering the behavior of a Lagrangian parcel that initially coincides with a small sector of the ring, see Panel B of Fig. 3. Since all other sectors of the ring will evolve in the same manner as that particular parcel given the axial symmetry of the system, the ratio of $\Omega(t)/\Omega(0)$ as determined from multiple sectors is the same as $\omega(t)/\omega(0)$ determined from a single sector.] Given this background, and using the conclusion from (b), would the azimuthal velocity of the ring intensify or weaken over time as the ring shrinks toward the sinkhole?

(d) Qualitatively, interpret the phenomenon described in (c) (i.e., the intensification or weakening of the vortex motion) in the context of *conservation of angular momentum* of the vortex ring.

Remarks:

(i) The phenomenon described in (b)-(d) is reminiscent of the generation of a vortex when one unplugs a bathtub filled with a shallow layer of water. In the latter case, the gentle stirring of water at the initial time (caused by the action of unplugging the bathtub using one's hand) produces the initial vorticity, $\Omega(0)$. The answers to (b)-(d) would provide the key to explaining how this "embryonic" vortex evolves into a strong vortex in the presence of the mass sink. [Of course, this remark and the discussions in (b)-(d) depend on the idealization that S = constant in Eq. (1). The reality could be more complicated in that S is generally a function of the radial distance to the sinkhole.]

(ii) The assumption of a fixed free surface is generally unrealistic. If that restriction is relaxed, the Lagrangian conservative quantity (when $S = 0$) will no longer be ω but ω/h, where h is the depth of water as a function of x and y. Moreover, when h is not constant in space, the horizontal components of pressure gradient force are generally non-zero. Phenomena such as gravity waves (see Chapter 3 of textbook) can exist in this case.

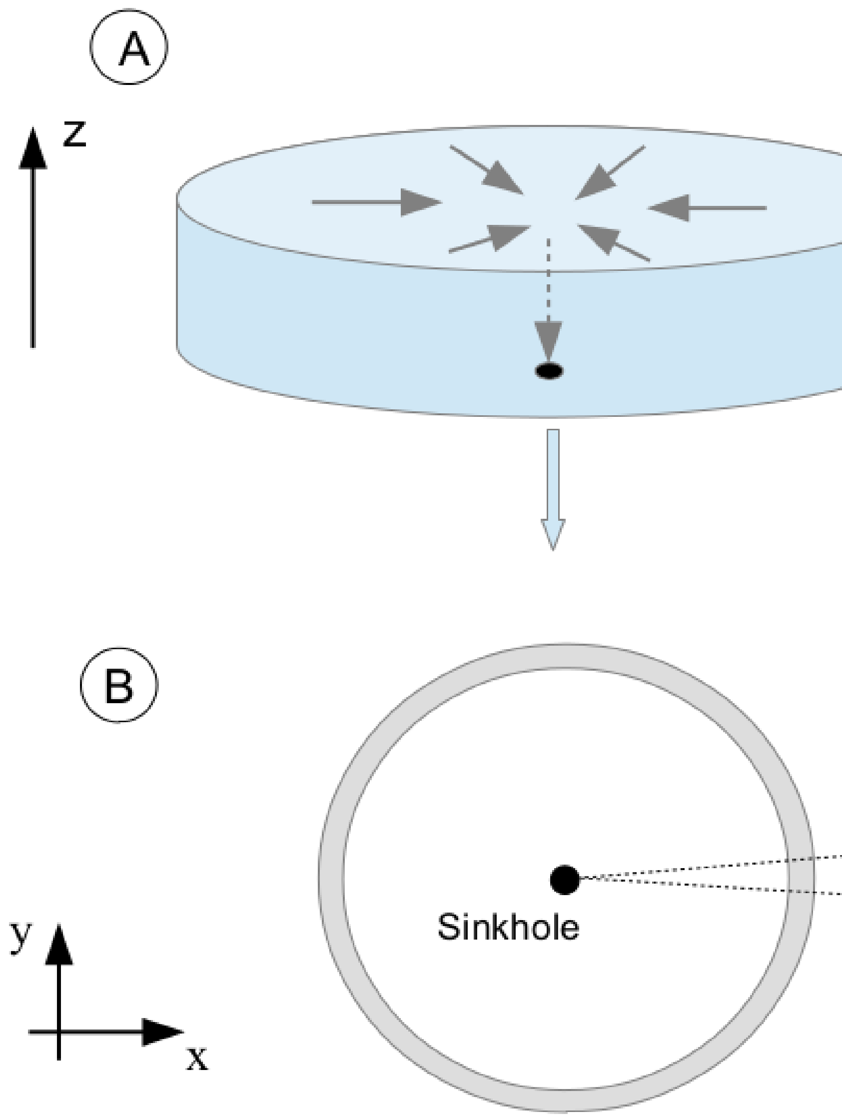


Fig. 3

Additional Note: For Prob. 1, the expected outcome will be a plot with the flow field and the outline of the "dyed blob" at different times. For example, if the velocity field in Prob. 1 is replaced by a pure deformation field with $u(x, y) = -x$, $v(x, y) = y$, and if we are asked to track the blob to $t = 1$, the plot would be as shown in Fig. 4. The circle filled in black is the initial blob given in Prob. 1. This plot was made by using **quiver** and **fill** commands of Matlab. The matlab code is provided below the plot.

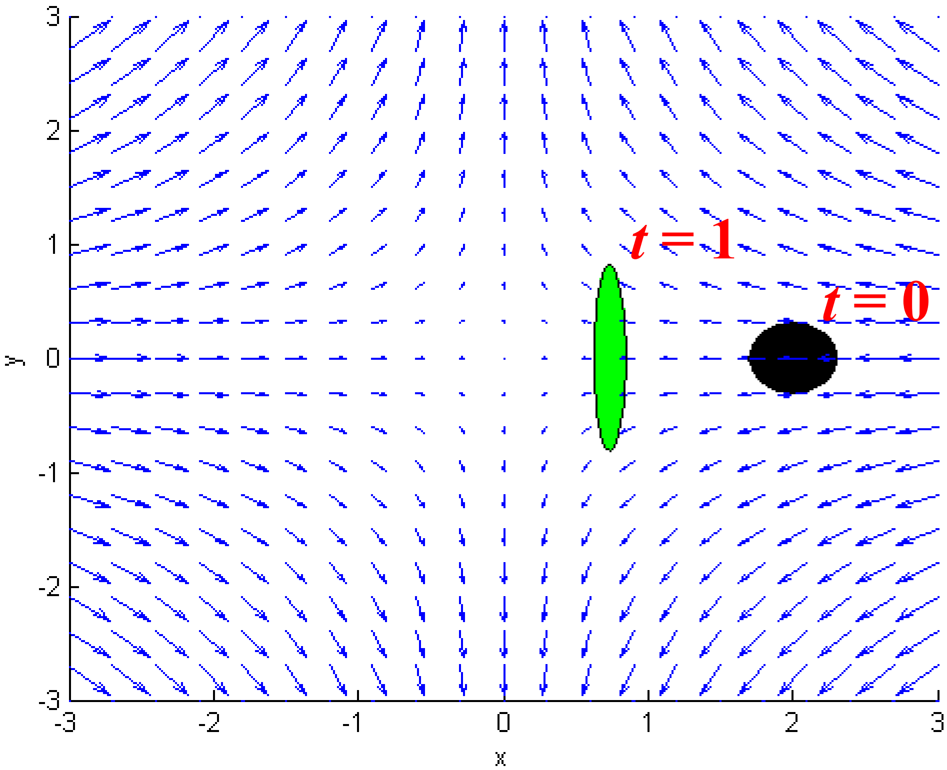


Fig. 4

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clear
x1d = [-3:0.3:3]; y1d = [-3:0.3:3];
for i = 1:length(x1d)
    for j = 1:length(y1d)
        x2d(i,j) = x1d(i); y2d(i,j) = y1d(j);
        u2d(i,j) = -x2d(i,j); v2d(i,j) = y2d(i,j);
    end
end
hold on
quiver(x2d,y2d,u2d,v2d,1)
axis([-3 3 -3 3])
N = 40; t = 1; dphi = 2*pi/N; r1 = 0.3
for n = 1:N
    phin = (n-1)*dphi;
    x0(n) = 2+r1*cos(phin); y0(n) = r1*sin(phin);
    xt(n) = x0(n)*exp(-t); yt(n) = y0(n)*exp(t);
end
fill(x0,y0, [0 0 0]);
fill(xt,yt, [0 1 0]);
```