MAE571, Fall 2016, Homework #4 (14 points)

Prob 1 (10 points)

Background: We have discussed Blasius' solution for a 2-D, steady, laminar boundary layer over a flat semi-infinite plate. As a quick summary (see Sec 8.3), with appropriate scaling for the boundary layer the governing equations for the 2-D steady flow are simplified into

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \, \frac{\partial^2 u}{\partial y^2} \quad , \qquad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad , \qquad (2)$$

with the boundary conditions (for $x \geq 0$),

$$u(x, 0) = 0 \ , \ \text{ and } \ u(x, y) \to U \ \text{ as } y \to \infty \ .$$

Here, $x = 0$ is the leading edge of the semi-infinite plate and U is the x-velocity of the impending flow which is uniform for $x < 0$. We seek the solution only over the domain of $x \geq 0$ and $0 \leq y < \infty$. From Eq. (2), defining stream function ψ by $(u, v) \equiv (\partial\psi/\partial y, -\partial\psi/\partial x)$, all terms in Eq. (1) can be expressed as a combination of ψ and its derivatives. Next, we seek a similarity solution of $u(\eta)$ where $\eta \equiv y/\mathrm{g}(x)$ and $\mathrm{g}(x) = (2\nu x/U)^{1/2}$. After a lengthy derivation (see Sec 8.3), the original PDE and boundary conditions are transformed to

$$\mathrm{f}''' + \mathrm{f}\,\mathrm{f}'' = 0 \ , \qquad (3)$$

with the boundary conditions,

(i) $\mathrm{f}(0) = 0$, (ii) $\mathrm{f}'(0) = 0$, (iii) $\mathrm{f}'(\eta) \to 1$ as $\eta \to \infty$,

where $(\ )' \equiv d(\ )/d\eta$, $\psi(x, y) = U\mathrm{g}(x)\mathrm{f}(\eta)$, and $u(\eta) = U\,\mathrm{f}'(\eta)$.

The tasks: Solve Eq. (3) with boundary conditions (i)-(iii) numerically to obtain $\mathrm{f}(\eta)$ and $\mathrm{f}'(\eta)$. (The shooting method, as discussed in class, is recommended but not required.) Use the solution to complete the following tasks. For tasks (b)-(d), set $\nu = 0.001$ (ν is kinematic viscosity) and $U = 1$.

(a) Plot the profile of $u(\eta)/U$ as a function of η. This is essentially a reproduction of Fig. 8.8.
(b) Plot v/U (v is the y-velocity) as a function of y at $x = 1$, 4, and 9. Please collect all three curves in one plot. For a comparison, make another plot of u/U as a function of y at $x = 1$, 4, and 9. Confirm that the assumption of $v \ll u$ (as used in deriving the equations for Blasius' solution) is sound.
(c) If the thickness of boundary layer at a given x, $\delta(x)$, is defined as the value of y at which $u(x,y) = 0.95\ U$, make a plot of $\delta(x)$ as a function of x.
(d) Blasius' solution reflects a perfect balance among the three terms in Eq. (1). To demonstrate that this is the case, plot the profiles of $A \equiv -u\partial u/\partial x$, $B \equiv -v\partial u/\partial y$, $C \equiv A+B$, and $D \equiv \nu\ \partial^2 u/\partial y^2$ (please collect all 4 curves in one plot) as a function of y at $x = 0.5$ and $x = 2$. Please use the same scale for the two plots to allow a clear comparison between them.

Prob 2 (4 points)

(a) Complete the exercise in Prob. 8.4 in the textbook.
(b) In addition, obtain the solution for the v-velocity of that problem. Given $M/\rho = 1$ and $\nu = 0.01$ (ν is kinematic viscosity), plot the profiles of the u-velocity and v-velocity as a function of y across the jet core, in the same fashion as Fig. 8.16, at $x = 1$ and 3.

[Note: This is an example of an "internal boundary layer". The flow inside the jet can be solved using the technique derived from classical boundary layer theory, yet there is no solid boundary associated with the "boundary layer".]