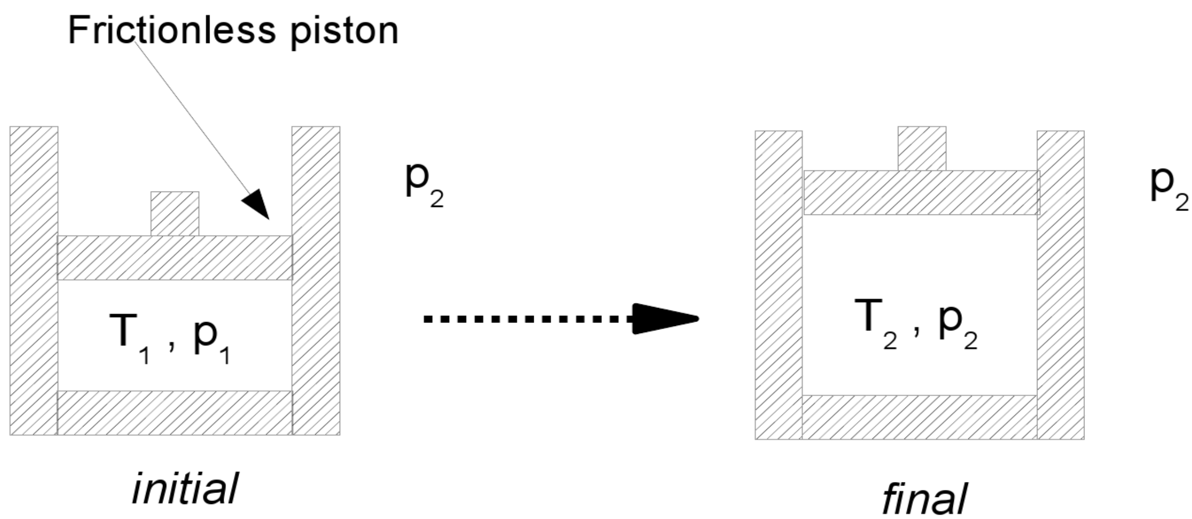
MAE578 Fall 2011 Homework 1 (Reviews on prerequisites)

3 points for everyone who returns the solutions on time, regardless of correctness of answers
1 point bonus if all questions are answered correctly
1 point ~ 1% of the total score for the semester

Thermodynamics

1. Consider an apparatus (see figure) that consists of an air-tight chamber capped by a frictionless piston. The chamber is filled with a diatomic ideal gas. Initially, the chamber is compressed such that the pressure inside it, P_1, is greater than the environmental pressure, P_2, with $P_1 = 1.3\ P_2$. The initial temperature inside the chamber is denoted as T_1. We then release the piston from its initial position and let the chamber freely expand until the pressure inside the chamber reaches that of the environmental pressure P_2. (At that point, the expansion will stop.) The apparatus is thermally insulated, i.e., no heat exchange takes place between the chamber and the outer environment. If the final temperature inside the chamber, T_2, is 25°C, what should be the initial temperature, T_1? (Ignore the weight of the piston.)



Hint and remarks: (1) While there are different approaches to solve this problem, the most straightforward is to consider that entropy (inside the chamber) is conserved in the process. (2) The outcome of this exercise is relevant to the process of convection and the determination of the vertical temperature profile for the atmosphere, as we will discuss later.

Multivariate calculus

2. A multivariate function is given as

$$u(p,q) = p\,q^3 + \exp(-q)$$

where p and q are themselves multivariate functions defined by $p(x,y) = 3x^2 + 4xy$, and $q(x,y) = y^2 + 3xy$. Evaluate $\partial u/\partial x$ and $\partial u/\partial y$ at $(x = 3, y = 2)$.

3. Given the vector **V** = (u, v, w) whose individual components are defined by

$u(x, y, z) = x - y^2 + 3z$
$v(x, y, z) = xy + yz$
$w(x, y, z) = x^2 + 4z$

(1) Obtain the expression of the divergence of **V**, $\nabla \bullet \mathbf{V}$, and evaluate $\nabla \bullet \mathbf{V}$ at the point of (x, y, z) = (1, 3, 2). Note that $\nabla \bullet \mathbf{V}$ is a scalar so you should obtain a single number.
(2) Obtain the expression of the curl of **V**, $\nabla \times \mathbf{V}$, and evaluate $\nabla \times \mathbf{V}$ at the point of (x, y, z) = (1, 3, 2). Note that $\nabla \times \mathbf{V}$ is a vector so you should obtain a vector with 3 elements.

Note: The exercise here will be relevant to a useful way to decompose a flow field. If **V** is the velocity of the fluid flow, $\nabla \times \mathbf{V}$ is what we call "vorticity". The significance of this variable for large-scale geophysical flows will be discussed later.

4. If u is a function of x and t and x itself is a function of t, the total derivative of u with respect to t can be derived as

$$du/dt = (\partial u/\partial x)(dx/dt) + \partial u/\partial t \,, \qquad \text{Eq. (1)}$$

where $\partial u/\partial x$ is the partial derivative with t held constant, and $\partial u/\partial t$ is the partial derivative with x held constant. Using this knowledge and given $x(t) = \sin(t) + t^2$ and $u(x,t) = x + x^2 t$, evaluate du/dt at x = 2, t = 1.

Remark: If the u in Eq. (1) happens to represent the velocity of a "fluid parcel" that moves along the x direction, where x itself is the distance along that direction and t is time, then we have $u \equiv dx/dt$. By this definition, Eq. (1) becomes

$$du/dt = u\,(\partial u/\partial x) + \partial u/\partial t \,.$$

We will later see that this provides a mathematical basis to transform the notion of conservation of momentum of a fluid parcel into a "field equation" for the fluid momentum in space and time. The latter provides a convenient form for computing the evolution of flow field (think weather prediction).

Conservation law

5. A specially constructed floor consists of a flat component and a slopping component that tilts at a 30° angle (see figure). Ignoring friction, if a block of metal is injected from the left with a 5 m/s initial speed, how far can it travel up the slope before it comes to a stop ? (Assume that this experiment is conducted at the surface of the Earth, $g = 9.8\ m/s^2$.)