Momentum, vorticity, and divergence

1. (a) Ignoring friction, if vertical velocity vanishes \((w = 0)\) for a certain fluid flow, the horizontal components of the momentum equation can be written as

\[
\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \frac{1}{\rho} \frac{\partial p}{\partial x},
\]

(1)

\[
\frac{\partial v}{\partial t} = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - \frac{1}{\rho} \frac{\partial p}{\partial y}.
\]

(2)

If the density of the flow is uniform in the horizontal direction, show that Eqs. (1) and (2) lead to

\[
\frac{d \zeta}{dt} = -\zeta D,
\]

(3)

where \(\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\) is the vorticity and \(D \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\) is the divergence of this flow. Therefore, if \(D \equiv 0\) (flow is "non-divergent") we have conservation of vorticity following the motion of a fluid parcel. [Note: When \(w = 0\) and \(\rho\) = constant, by the continuity equation \(D\) is guaranteed to be zero unless there is a mass source or sink.]

(b) In the case when \(D \neq 0\), Eq. (3) indicates that convergence \((D < 0)\) leads to an amplification of the vortex motion while divergence \((D > 0)\) leads to damping of vorticity. One can appreciate this behavior by momentarily holding \(D\) as a constant, which leads to \(\zeta(t) = \zeta(0) \exp(-D t)\). This behavior is also consistent with daily experience; If we unplug a bathtub filled with water, the mass loss through the sinkhole would momentarily create a convergence \((D < 0)\). Accompanying it, we see an amplification of the vortex motion surrounding the sinkhole. Now that we have a mathematical basis in Eq. (3), try to physically interpret this phenomenon. [3 points]

Vertical motion and thermodynamic equation

2. Solve Prob 12 of Chapter 4. [2 points]

Rotating frame; Coriolis and centrifugal forces

3. Solve Prob 3(a) of Chapter 6. (You do not have to solve Part (b) of that problem.) [1 point]

4. Solve Prob 5 of Chapter 6. You do not have to answer the last question: "What analogies can you draw...?" [1 point]

Geostrophic balance

5. Solve Prob 6 of Chapter 7. [2 points]