MAE578 Fall 2011 Homework #6

Momentum, vorticity, and divergence

1. (a) Ignoring friction, if vertical velocity vanishes (w = 0) for a certain fluid flow, the horizontal components of the momentum equation can be written as

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad , \tag{1}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad . \tag{2}$$

If the density of the flow is uniform in the horizontal direction, show that Eqs. (1) and (2) lead to

$$\frac{d\zeta}{dt} = -\zeta D \quad , \tag{3}$$

where $\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the *vorticity* and $D \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is the *divergence* of this flow. Therefore, if $D \equiv 0$ (flow is "non-divergent") we have *conservation of vorticity following the motion of a fluid parcel*. [Note: When w = 0 and ρ = constant, by the continuity equation D is guaranteed to be zero unless there is a mass source or sink.]

(b) In the case when $D \neq 0$, Eq. (3) indicates that convergence ($D < 0$) leads to an amplification of the vortex motion while divergence ($D > 0$) leads to damping of vorticity. One can appreciate this behavior by momentarily holding D as a constant, which leads to $\zeta(t) = \zeta(0)\exp(-D\,t)$. This behavior is also consistent with daily experience; If we unplug a bathtub filled with water, the mass loss through the sinkhole would momentarily create a convergence ($D < 0$). Accompanying it, we see an amplification of the vortex motion surrounding the sinkhole. Now that we have a mathematical basis in Eq. (3), try to physically interpret this phenomenon. **[3 points]**

Vertical motion and thermodynamic equation

2. Solve Prob 12 of Chapter 4. **[2 points]**

Rotating frame; Coriolis and centrifugal forces

3. Solve Prob 3(a) of Chapter 6. (You do not have to solve Part (b) of that problem.) **[1 point]**

4. Solve Prob 5 of Chapter 6. You do not have to answer the last question: "What analogies can you draw...?" **[1 point]**

Geostrophic balance

5. Solve Prob 6 of Chapter 7. **[2 points]**