Convection and buoyancy oscillation

Recap:

We analyzed the static stability of a vertical profile by the "parcel method"; For a given environmental profile (of T_0, p_0, θ_0, etc.), if the density of an air parcel that's been carved out of the environment at z and lifted to $z+\Delta z$ is less than the density of the environment at $z+\Delta z$, we have static instability. By this argument, we were able to determine that the criterion for static stability is $d\theta_0 /dz > 0$ (and instability $d\theta_0 /dz < 0$).

We can extend the analysis to quantify the positive/negative buoyancy experienced by the air parcel in the stable or unstable cases.

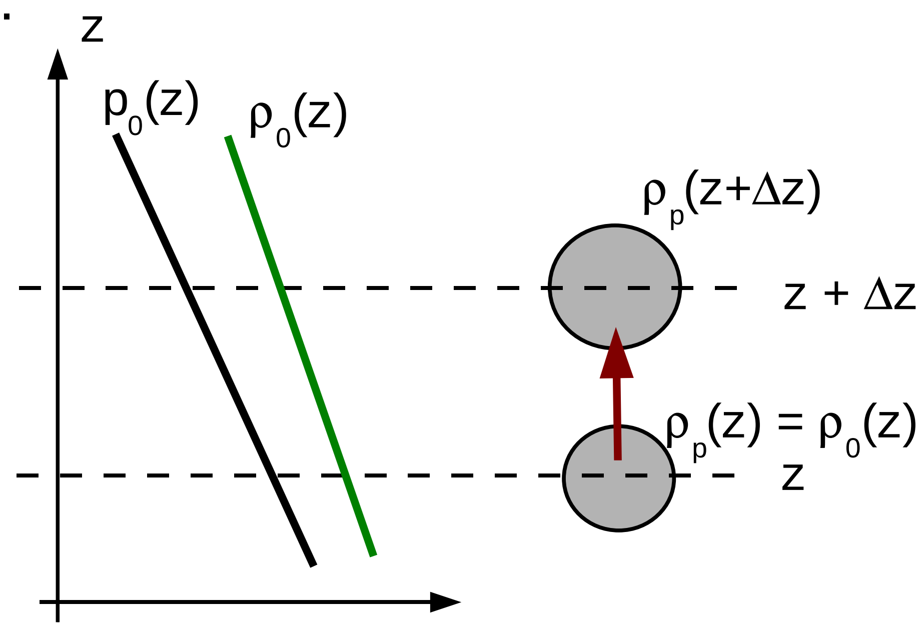
As before, we will use subscript "0" to denote an environmental variable and "p" to denote a variable following the air parcel.

General argument (works for both liquid fluid and ideal gas)

The environment (and the air parcel) at level z is in hydrostatic balance:
$(dp_0/dz)_{\text{at } z} = -\ \rho_0(z)\ g\ \ = -\ \rho_p(z)\ g$

The environment at level z+Δz is also in hydrostatic balance:
$(dp_0/dz)_{\text{at } z+\Delta z} = -\ \rho_0(z+\Delta z)\ g$,

For the air parcel at z+Δz, the net upward force is just the environmental pressure gradient, $(dp_0/dz)_{\text{at } z+\Delta z}$ (recall that the pressure of the parcel is the same as the pressure of the environment). The downward force acting on the parcel is $-\rho_p(z+\Delta z)\ g$. Since $\rho_p(z+\Delta z) \neq \rho_0(z+\Delta z)$, we now have an imbalance of the net force acting on the parcel.



(continued)

The imbalance of force (acting on the parcel) at $z+\Delta z$ would lead to acceleration. Bringing back Newton's law of motion, we have

$$\rho_p(z+\Delta z)\ dw/dt = -(dp_0/dz)_{\text{at } z+\Delta z} - \rho_p(z+\Delta z)\ g \ , \qquad \text{Eq. (1)}$$

where w is the vertical velocity of the parcel at $z+\Delta z$ (so dw/dt is the acceleration of the parcel). Eq. (1) is the basic equation of motion for the air parcel at $z+\Delta z$.

Since $(dp_0/dz)_{\text{at } z+\Delta z} = - \rho_0(z+\Delta z)\ g$, Eq. (1) can be rewritten as

$$dw/dt = g\ [\rho_0(z+\Delta z) - \rho_p(z+\Delta z)]/\rho_p(z+\Delta z) \qquad \text{Eq. (2)}$$

We may define the r.h.s. of Eq. (2) as the buoyancy experienced by the parcel, $b(z+\Delta z)$, then $dw/dt = b$.

Case 1: Liquid (nearly incompressible) fluid

In this case, since the density of the parcel is nearly conserved, we have $\rho_p(z+\Delta z) = \rho_p(z) = \rho_0(z)$

$\Rightarrow$ Eq. (2) becomes

$$dw/dt = g\,[\rho_0(z+\Delta z) - \rho_0(z)]/\rho_0(z)$$

If Δz is small, Taylor series expansion leads to

$$\rho_0(z+\Delta z) \approx \rho_0(z) + (d\rho_0/dz)\,\Delta z,$$

so we have

$$dw/dt = \{g\,(d\rho_0/dz)/\rho_0\}\,\Delta z$$

Noting that $w \equiv d(\Delta z)/dt$ (vertical velocity is the time derivative of the vertical displacement of the parcel), and define

$$N^2 \equiv -\{g\,(d\rho_0/dz)/\rho_0\}\,, \qquad \text{Eq. (3)}$$

we have

$$d^2(\Delta z)/dt^2 = -N^2\,(\Delta z)\,. \qquad \text{Eq. (4)}$$

(continued)

If $N^2 > 0$, the solution of Eq. (4) is $\Delta z(t) = A \sin(Nt) + B \cos(Nt)$; The vertical displacement of the parcel oscillates in time $\Rightarrow$ stable

In this case, $N = [- \{g (d\rho_0/dz)/\rho_0\}]^{1/2}$ is the Brunt-Väisälä frequency (or buoyancy frequency); The period of oscillation is $\tau = 2\pi/N$.

If $N^2 < 0$, the solution of Eq. (4) is $\Delta z(t) = A \exp(Nt) + B \exp(-Nt)$; The vertical displacement of the parcel will "run away" $\Rightarrow$ unstable

By the definition of N^2, "$N^2 > 0$" corresponds to $d\rho_0/dz < 0$. ***Indeed, a liquid (nearly incompressible) fluid is expected to be statically stable when the density of the fluid decreases with height.***

Case 2: The atmosphere

As discussed before, potential temperature is now a more useful variable to consider. Recall Eq. (2):

$$dw/dt = g\,[\rho_0(z+\Delta z) - \rho_p(z+\Delta z)]/\rho_p(z+\Delta z)\ , \qquad \text{Eq. (2)}$$

which is valid whether the fluid is liquid or an ideal gas.

1. Note that the parcel that's been lifted to $z+\Delta z$ has the same pressure as the environment at $z+\Delta z$; $p_0(z+\Delta z) = p_p(z+\Delta z) = p^*$ (p^* is a certain constant). Using ideal gas law, we have

$$\rho_0(z+\Delta z) = (p^*/R)\,[1/T_0(z+\Delta z)]\ ,\quad \rho_p(z+\Delta z) = (p^*/R)\,[1/T_p(z+\Delta z)]\ ,$$

therefore Eq. (2) can be rewritten as

$$dw/dt = g\,[T_p(z+\Delta z) - T_0(z+\Delta z)]/T_p(z+\Delta z) \qquad \text{Eq. (5)}$$

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(continued)

2. From the definition of potential temperature, we have

$$\theta_0(z+\Delta z) = C\,T_0(z+\Delta z)\ ,\ \ \theta_p(z+\Delta z) = C\,T_p(z+\Delta z)\ ,\ C \equiv (p_S/p^*)^{R/Cp}$$

(we have again used $p_0(z+\Delta z) = p_p(z+\Delta z) = p^*$). Equation (5) can then be rewritten as

$$dw/dt = g\,[\theta_p(z+\Delta z) - \theta_0(z+\Delta z)]/\theta_p(z+\Delta z)\ . \qquad \text{Eq. (6)}$$

We have gone the distance to transform Eq. (2) to Eq. (6) so we can now incorporate conservation of potential temperature (for the parcel that undergoes an adiabatic process) to turn the r.h.s. into a function of the environmental variables only

Since $\theta_p(z+\Delta z) = \theta_p(z) = \theta_0(z)$, Eq. (6) becomes

$$dw/dt = g\,[\theta_0(z) - \theta_0(z+\Delta z)]/\theta_p(z) \qquad \text{Eq. (7)}$$

(continued)

Again, we can define the r.h.s. of Eq. (7) as the buoyancy, b, at $z+\Delta z$.

Invoking Taylor series expansion, $\theta_0(z+\Delta z) \approx \theta_0(z) + (d\theta_0/dz)\,\Delta z$, and replacing w by $d(\Delta z)/dt$, Eq. (7) becomes

$$d^2(\Delta z)/dt^2 = -N^2\,(\Delta z)\,, \qquad \text{Eq. (8)}$$

with

$$N^2 \equiv g\,(d\theta_0/dz)/\theta_0\,. \qquad \text{Eq. (9)}$$

We have stability (parcel oscillates vertically) if $N^2 > 0$, instability (parcel keeps rising) if $N^2 < 0$.

The Δz in Eq. (8) can be either positive (upward perturbation) or negative (downward perturbation). If the vertical profile is unstable and we start with a downward perturbation, the parcel will keep sinking, and so on.

In Case 2, the stable situation, $N^2 > 0$, corresponds to $d\theta_0/dz > 0$, which is consistent with our previous qualitative argument (see previous set of slides).

Beware that in Case 1, the condition for stability is that density of the incompressible fluid *decreases* with height. In Case 2, the stability condition for the atmosphere is that potential temperature *increases* with height.

Note as well that in Eq. (2) we have $\rho_0(z+\Delta z) - \rho_p(z+\Delta z)$ (environment minus parcel) in the numerator for buoyancy, while in Eq. (6) we have $\theta_p(z+\Delta z) - \theta_0(z+\Delta z)$ (parcel minus environment) in the numerator. Without a detailed proof, we should mention that under some further assumptions one can indeed obtain an approximation for the atmosphere as

$$\frac{\rho'}{\bar{\rho}} \approx -\frac{\theta'}{\bar{\theta}} \ ,$$

where the prime indicates "perturbation" and bar indicates "mean" (we should define them later).

Reasons why we like potential temperature:

(1) θ is a proxy of entropy; A parcel that undergoes an adiabatic expansion or contraction conserves θ, just like it conserves entropy S.

(2) θ has the unit of temperature and can be physically related to temperature following an adiabatic ascent or descent of the air parcel.

(3) The "perturbation" of potential temperature, θ', is approximately proportional to (the negative of) the perturbation of density, $-\rho'$. Therefore, for various applications we could also regard θ as a proxy of density.

Potential temperature as a vertical coordinate

Previously, we noted that under hydrostatic balance pressure increases monotonically with height

$\Rightarrow$ One-to-one correspondence between p and z

$\Rightarrow$ "Pressure" can replace "height" as an alternative vertical coordinate

if an atmosphere is statically stable, $d\theta_0/dz > 0$ everywhere, then there is also a one-to-one correspondence between θ_0 and z

$\Rightarrow$ One may use potential temperature as an alternative vertical coordinate

As before, one can apply chain rule to do the coordinate transformation. For example,

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial z} \;,$$

and so on.