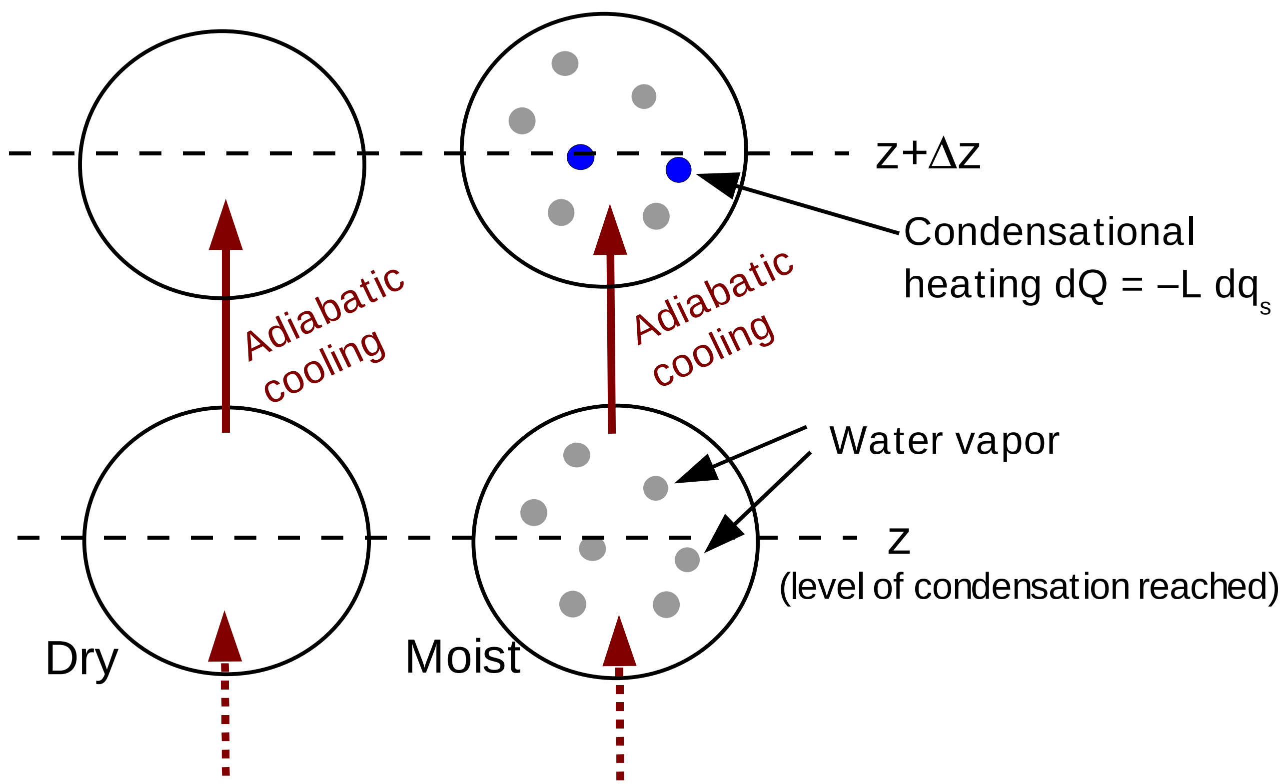
Equivalent potential temperature

Recall that $dQ = C_P\, dT - \alpha\, dp = C_P\, T\, d \ln \theta$. For a dry air parcel undergoing an adiabatic ascent, $dQ = 0 \Rightarrow d\theta = 0$, potential temperature is conserved.

For a moist air parcel, before condensation occurs, we still have $dQ = 0$ and $d\theta = 0$. Once condensation occurs (when the parcel is lifted to a level with sufficiently cold temperature), by latent heat release we will have $dQ > 0$ for the air parcel $\Rightarrow \theta$ is no longer conserved.



Saturated moist adiabatic process ; latent heat release

Note that q_s generally decreases with temperature. If q_s decreases by the amount of Δq_s as an air parcel is lifted from z to z+Δz, the amount of water vapor that has to condense into liquid water is $\Delta m_v = m_{tot}\, \Delta q_s$ (recall that $q \equiv m_v/m_{tot}$, where m_v is the mass of water vapor and m_{tot} is the mass of the whole air parcel; furthermore, $q = q_s$ since the air is saturated). The amount of latent heat release due to condensation is $\Delta Q^* = - L\, \Delta m_v$, where L is the latent heat constant (for condensation). We have a negative sign here because we define a positive ΔQ^* as heating for the air parcel. Note that Δm_v is negative (the amount of water vapor decreases as q_s decreases) so ΔQ^* is positive. Alternatively, we can state that $\Delta Q^* = L\, \Delta m_w$, where Δm_w is the gain of the mass of liquid water.

The heating per unit mass for the air parcel is $\Delta Q = \Delta Q^*/m_{tot} = -L\, \Delta q_s$
In differential form, we have

$$dQ = -L\, dq_s$$

We may freely exchange q and q_s as long as the parcel remains saturated.

We therefore arrive at the modified thermodynamic equation for the saturated moist adiabatic process,

$$-L\, dq_s = C_P\, T\, d \ln \theta \qquad \text{Eq. (1)}$$

or $$-(L/C_P)\, dq_s/T = d \ln \theta \qquad \text{Eq. (1a)}$$

Although L and C_P varies slightly with temperature, we keep them constant for simplicity. ($L \approx 2.5 \times 10^6$ J kg^{-1} at 0 °C)

Given that $\frac{d\,q_s}{q_s} \gg \frac{d\,T}{T}$ under a wide range of conditions (see previous set of slides), we have $d(q_s/T) = (q_s/T)\,(dq_s/q_s - dT/T) \approx (q_s/T)\,(dq_s/q_s) = (dq_s)/T$, so Eq. (1a) can be approximated by

$$-d(Lq_s/C_P T) = d \ln \theta ,$$

or

$$d \ln [\, \theta \exp(Lq_s/C_P T)\,] = 0$$

Define equivalent potential temperature as

$$\theta_e \equiv \theta \exp(Lq_s/C_P T) ,$$

we then have

$$d \ln \theta_e = 0 ,$$

i.e., **θ_e is conserved following the saturated moist adiabatic ascent of an air parcel**.

Saturated adiabatic lapse rate

Following the ascent of a saturated air parcel, we have

$$-L\, dq_s = C_P\, T\, d \ln \theta \, . \qquad \text{Eq. (1) (repeat)}$$

Here, it is understood that the q_s,T and θ are the quantities associated with the air parcel. We will omit the subscript "p" used in our previous discussion for dry adiabatic lapse rate. Also, recall that $d(\ln \theta)/dz = T^{-1}(dT/dz + \Gamma_d)$, where $\Gamma_d = g/C_P$ is the dry adiabatic lapse rate. We will now define $\Gamma_s \equiv - dT/dz$, where the dT/dz here is the rate of change of temperature following the parcel. our goal is to uncover the relation between Γ_s and Γ_d. Since q_s depends on T and p (recall that $q_s \approx \varepsilon\, e_s/p$ where e_s depends on T) , we have

$$\frac{dq_s}{dz} = \frac{\partial q_s}{\partial T}\frac{dT}{dz} + \frac{\partial q_s}{\partial p}\frac{dp}{dz}$$

$$= -\frac{\varepsilon}{p}\frac{\partial e_s}{\partial T}\Gamma_s + \varepsilon e_s p^{-2} \rho g \qquad \text{Step (1)}$$

$$= -\frac{d\ln(e_s)}{dT} q_s \Gamma_s + \frac{g}{RT} q_s \qquad \text{Step (2)}$$

$$= -\beta \Gamma_s q_s + \frac{C_p}{RT}\Gamma_d q_s \qquad \text{Eq. (2)}$$

In the above derivation, we have used the hydrostatic equation in step (1), ideal gas law in step (2), and the approximate form of Clausius-Clapeyron equation, $e_s(T) = A \exp(\beta T)$ (as in M&P textbook) in the last step in obtaining Eq. (2). Although in the formula for $e_s(T)$ the "T" is in °C, we can replace it by a T in °K and still obtain the relation of $d \ln(e_s)/dT = \beta$. The gas constant "R" in Eq. (2) is for the mixture of dry air and water vapor. It deviates slightly from the gas constant for dry air, R_d, but this is not critical for our discussion.

Using Eq. (2), Eq. (1) can be rewritten as

$$-\frac{L q_s}{C_p T}\left(\frac{C_p}{R T}\Gamma_d q_s - \beta \Gamma_s q_s\right) = \frac{1}{T}(\Gamma_d - \Gamma_s) \ ,$$

or

$$\Gamma_s = \Gamma_d \times \frac{1+\frac{L q_s}{R T}}{1+\frac{\beta L q_s}{C_p}} \ . \qquad \text{Eq. (3)}$$

This is Eq. (4-28) in M&P. The factor in the r.h.s. is generally less than 1, so $\Gamma_s < \Gamma_d$. This is solely expected since latent heat release by condensation would compensate adiabatic cooling, such that a saturated moist air parcel would not cool as rapidly as a dry parcel as they are lifted upward by the same distance.

Finally, we can determine the static stability of the environmental profile by comparing the environmental lapse rate with Γ_s.

For dry air

$\Gamma < \Gamma_d$: Stable

$\Gamma > \Gamma_d$: Unstable

For saturated moist air

$\Gamma < \Gamma_s$: Stable

$\Gamma > \Gamma_s$: Unstable

$$\Gamma_d \equiv -dT_p/dz = g/C_P \approx 10\ °K/km$$

$\Gamma_s < \Gamma_d$, Γ_s can vary from 3 to close to 10 °K/km

An environmental profile, T(z), could be stable in the dry sense but still unstable in the moist saturated sense. (See illustration below.) Physically, this is because latent heat release provides extra buoyancy for the air parcel to continue its ascent. Therefore, the existence of water vapor in the atmosphere generally help enhance convection.

Moist convection is usually more energetic than dry convection.