Stefan-Boltzmann law

Recall Planck function

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5(\exp^{[\frac{hc}{\lambda kT}]} - 1)}$$ Eq. (A-1) in M&P

h = Planck's constant, k = Boltzmann's constant, c = speed of light, T = temperature (in degK)
$B_\lambda(T)$ is a function of the "wavelength" λ (of electromagnetic waves)

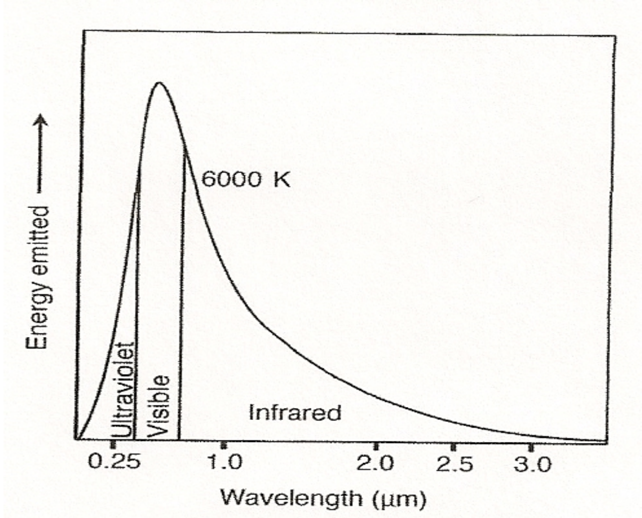


Fig. 2.2 in M&P

The total (rate of) energy output is the integral of $B_\lambda(T)$ over λ.
(More precisely, there's an additional factor of π; see next slide)

A quick technical point:

The $B_\lambda(T)$ in previous slide is the "monochromatic radiation intensity" (per radian solid angle). The "monochromatic irradiance (or radiative energy flux)" is $\pi B_\lambda(T)$, provided that the radiation is isotropic (independent of direction).

Some extra detail:

Think of a small area element at the surface of the sun: It emits radiation with the intensity of $B_\lambda(T)$ in all direction, thereby producing a total energy flux of $4\pi B_\lambda(T)$, where 4π is the total solid angle of a sphere enclosing that area element. Note, however, that a half of this energy flux goes back into the interior of the sun. Of the other half that comes out of the surface, we are only concerned with the component that is normal to the surface (which is the radiation that will go out to space for the whole universe to see). In mathematical terms, the relevant radiative energy flux is

$$\int_\Omega B_\lambda \cos(\phi)\, d\Omega = \pi B_\lambda \ .$$

Here, the integral is over the half sphere that raises above the surface of the sun, Ω is solid angle, and ϕ is the angle between a ray of radiation emitted by the small area element and the upward-pointing vector normal to the surface of the sun.

No need to worry about this detail. For our purpose, it suffices to take the extra factor of π as given and move on. Interested students can consult any textbook on atmospheric radiation.

For a blackbody with surface temperature *T*, the energy flux (per unit surface area) emitted by that object is

$$\pi B(T) \equiv \int_0^\infty \pi B_\lambda(T)\, d\lambda = \sigma T^4 \ ,$$

$$\sigma = 2\pi^5 k^4 / 15 h^3 c^2 \qquad \text{Eq. (A-2) in M\&P}$$

This is known as Stefan-Boltzmann law, which states that
the rate of outward radiative energy (per unit area) emitted by an object with temperature *T* is proportional to the **4th power of *T***

The higher the temperature of an object, the greater its radiative energy output will be

The Stefan-Boltzmann constant $\sigma \approx 5.67 \times 10^{-8}$ W m^{-2} K^{-4}

$\pi B(T)$ has the unit of W m^{-2}

The total outward radiative energy output of an object with surface temperature *T* and surface area *A* is $\pi B(T) \times A$

The Sun

Radius of the sun $R_S \approx 696000$ km, or 696000000 m

Surface area of the sun, $A_S = 4 \pi R_S^2 = 6.08 \times 10^{18}$ m^2

Outward radiative energy flux (per unit area) at the sun's surface ($T \approx 6000$K) is
$\pi B(T) = \sigma T^4 = (5.67 \times 10^{-8}) \times (6000)^4 = 7.35 \times 10^7$ W m^{-2}

The total radiative energy output of the Sun is

$$\pi B(T) \times A_S \approx 4.47 \times 10^{26} \text{ W}$$

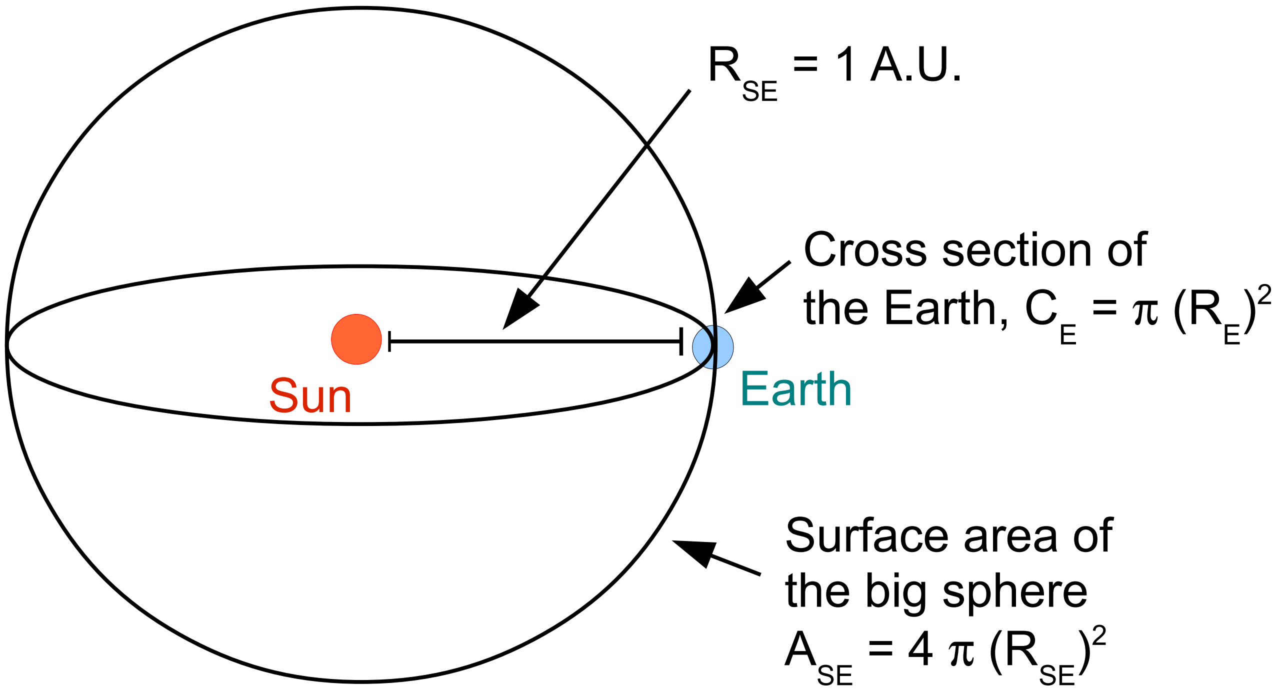
The actual temperature at the surface of the sun is slightly less than 6000K. This gives rise to the slightly smaller value of 3.87×10^{26} W in p. 9 of M&P textbook. We will follow the textbook by adopting this value.

Every second, the Sun emits 3.87×10^{26} Joule of energy into the Universe. How much of it is intercepted by the Earth?

The distance between the Sun and the Earth is R_{SE} = 1 A.U. = 150,000,000 km
The big sphere centered at the Sun with a radius of R_{SE} (see figure) has the total surface area of $A_{SE} = 4\pi (R_{SE})^2$. The energy output of the sun, 3.87×10^{26} W, is uniformly distributed on this sphere.

The cross section (that intercepts sun light) of the Earth is $C_E = \pi (R_E)^2$. where R_E = 6370 km is earth's radius. (See next slide)

=> The energy intercepted by the Earth is $(3.87 \times 10^{26}\ \text{W}) \times (C_E/A_{SE}) = 1.74 \times 10^{17}$ W



(The cross section of the Earth, $C_E = \pi (R_E)^2$)

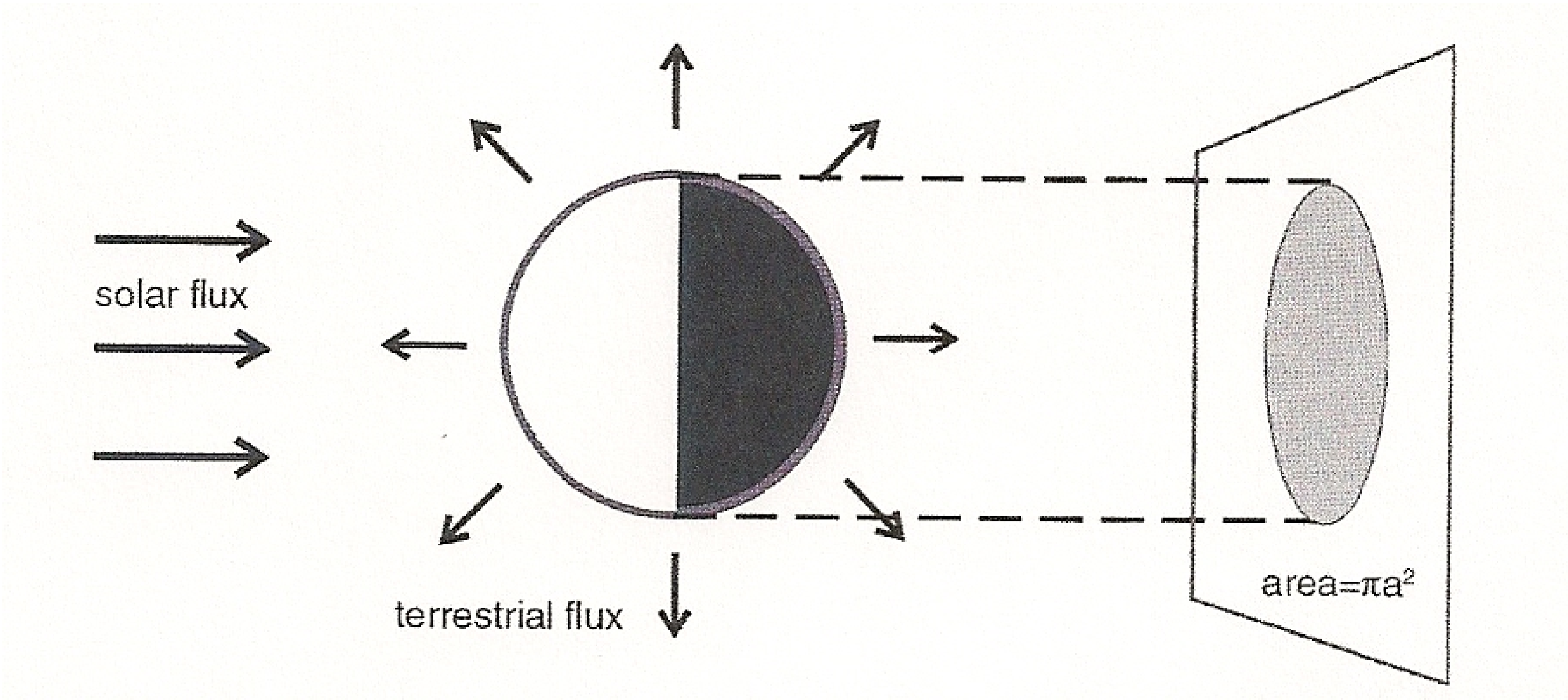
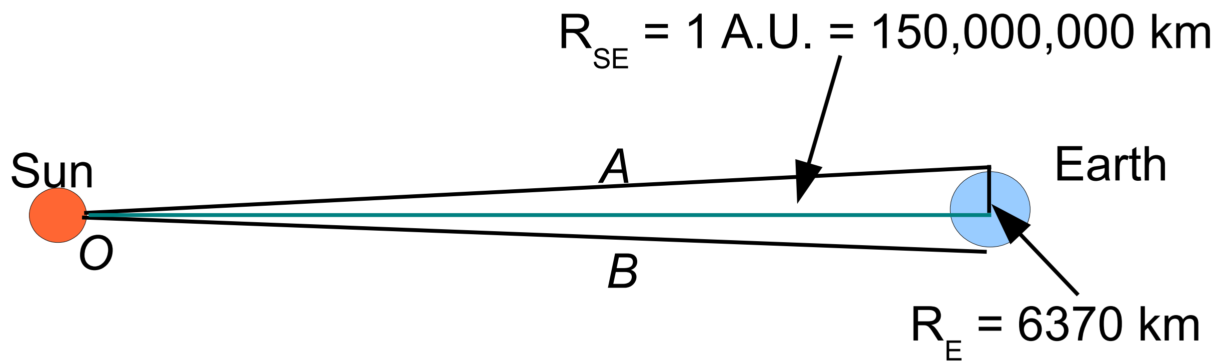


Fig. 2.4 in M&P

It is justifiable to assume that the sunlight that reaches the Earth is parallel. See next slide.

"Plane Parallel" Assumption

The sunlight that reaches the Earth can be considered "plane parallel" because of the large distance between the Sun and the Earth, and because of smallness of the Earth itself.



angle $AOB = 2\ \theta$, where $\tan\theta = R_E/R_{SE}$

Since $\tan\theta \ll 1$, $\theta \sim \tan\theta$

==> angle $AOB \sim 2\ R_E/R_{SE}$ = 0.000085 radian = 0.0048 degree

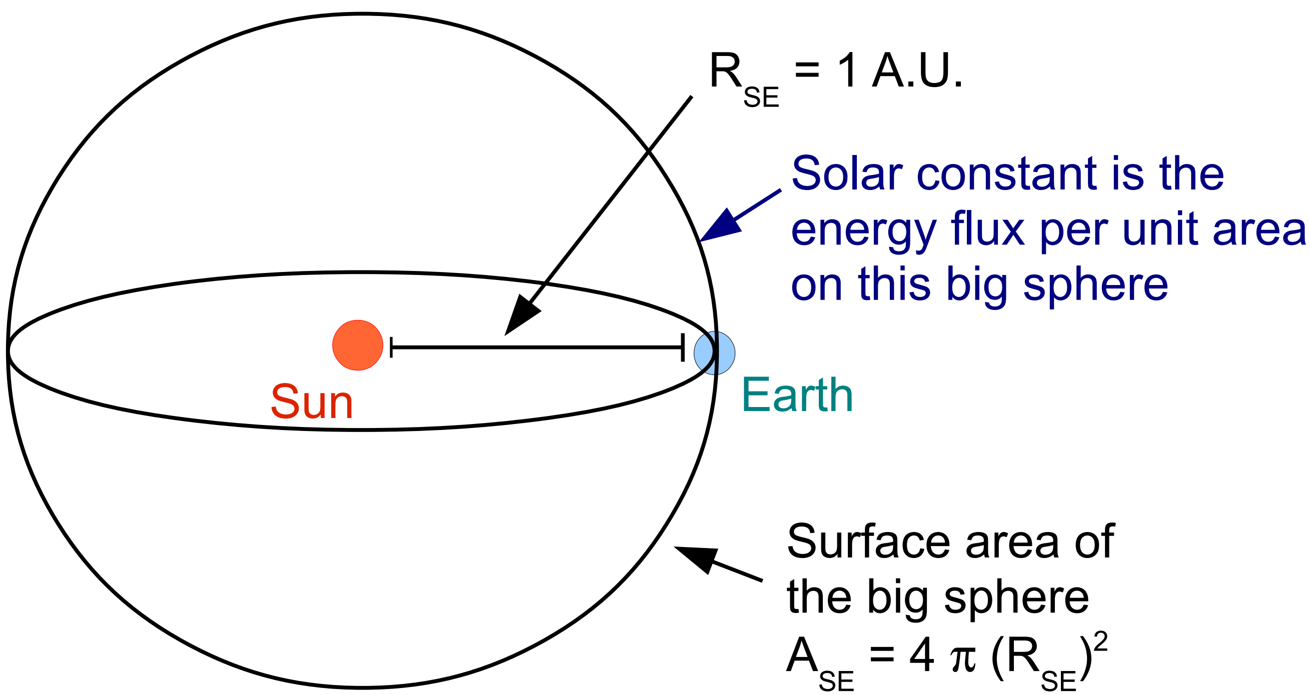
==> Rays *OA* and *OB* are almost parallel

(Note: $\tan\theta = \theta + \theta^3/3 + 2\ \theta^5/15 + ...$)

Solar constant

It is often more convenient to consider the radiative energy *per cross-sectional unit area* that the Sun provides for the Earth system. This quantity is called the "Solar constant" and is simply the total energy output of the sun divided by the the area (A_{SE}) of the "big sphere",

$$S_o = (3.87 \times 10^{26}\ W)/A_{SE} = 1368\ W\ m^{-2}$$



Multiply S_o by the cross section of the Earth, C_E, we recover the total energy intercepted by the Earth as $S_o \times C_E = 1.74 \times 10^{17}$ W. Recall that $C_E = \pi (R_E)^2$.

If we redistribute this amount of energy uniformly to the surface of the Earth, with its area being $A_E = 4 \pi (R_E)^2$, the energy per unit area of solar radiation received by the Earth would be

$$S_o \times C_E/A_E = S_o/4 = 342 \text{ W m}^{-2}$$

Note:
(1) This is assuming that the tropics and polar regions receive the same energy per unit area. In reality, the tropics receives more energy of course.

(2) In determining the solar constant S_o or $S_o/4$, we have lumped together day and night and the four seasons; Note that Earth's cross sectional area never changes regardless of time of the day or day of the year. Therefore, The amount of solar energy intercepted by the Earth is the same all the time.

(S_o does vary slightly with season because Earth's orbit is slightly elliptical. In the present epoch, Earth is slightly closer to the Sun during Northern Hemisphere winter. Let's not worry about such detail for now.)

As an aside, the value of $S_o/4 = 342$ W m^{-2} is useful to remember at least for its "order of magnitude" when one thinks of solar energy applications. (A typical solar panel for household uses has the size of a few m^2. Its efficiency is however much less than 100%.)

Note that the value, $S_o/4 = 342$ W m^{-2}, is the average over day and night, winter and summer, and all latitudes. If you live on a tropical island, the local value would be higher.

Part of the solar radiative energy from $S_o/4 = 342$ W m^{-2} is **bounced back to space by reflection of Earth's surface and clouds** (and to a lesser extent by the scattering by air molecules - this process is what makes the sky blue)

If a fraction, α_p, of the solar radiative energy is bounced back to space, the net per-unit-area solar energy received by the Earth would be only

$$(S_o/4)\,(1 - \alpha_p)$$

We call α_p the "planetary albedo", nominally defined as the global average of the reflectivity of the Earth system.   We have some observations (mainly by satellites) to estimate this quantity.

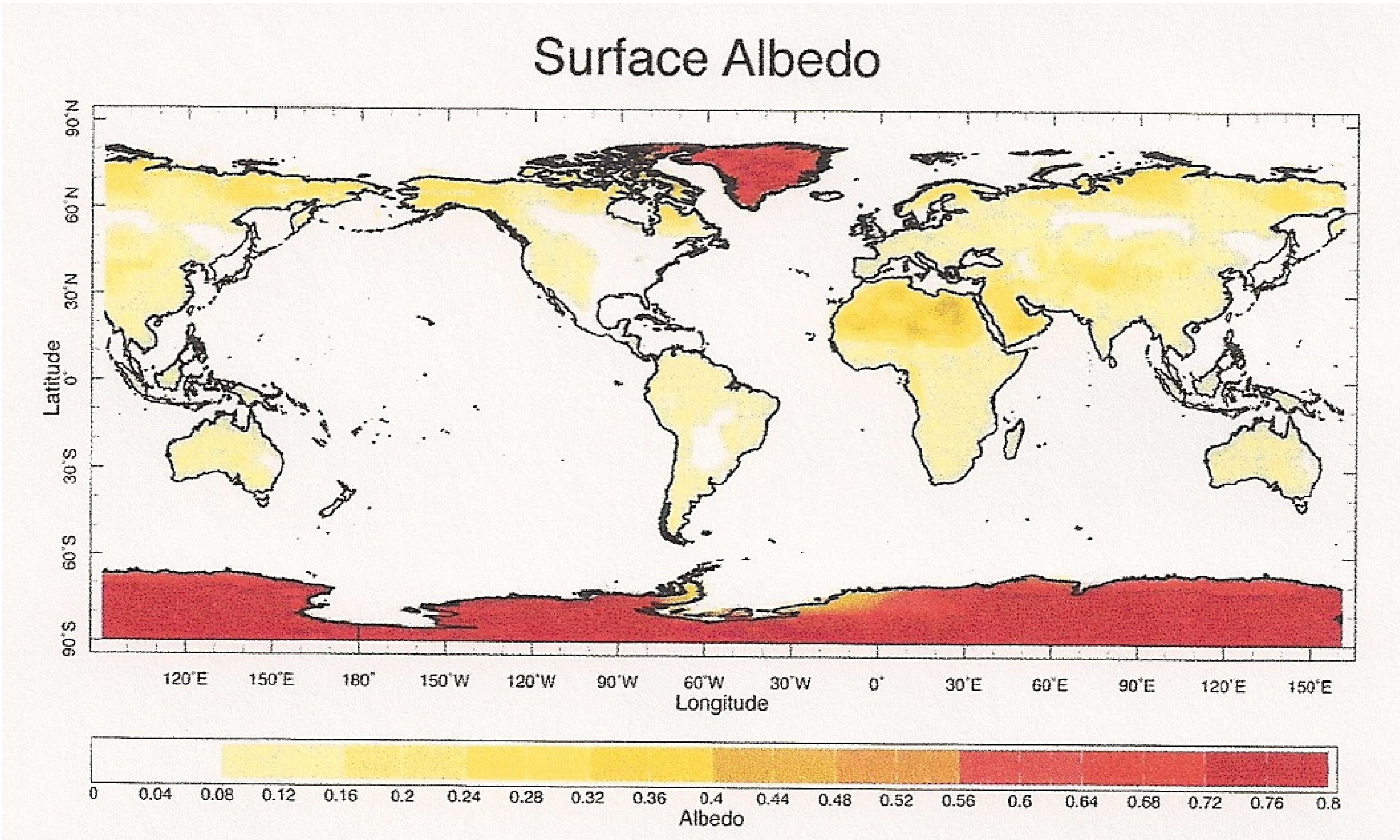


Fig. 2.5 in M&P (Ocean has small albedo; looks "dark" in satellite photos)

Earth in radiative equilibrium

If Earth keeps receiving energy from the Sun without giving it back (to space), it will become hotter and hotter. But recall that any object that has a non-zero temperature emits radiation. As Earth gets hotter, it emits more energy back to space. At a certain temperature T_E, an equilibrium will be reached such that the solar energy received by the Earth equals the energy that the Earth radiates back to space.

The incoming solar energy per unit area is $(S_o/4)\,(1 - \alpha_p)$

At temperature T_E, the energy per unit area radiated by the Earth is (by Stefan-Boltzmann law) $\sigma\, T_E^4$

Let "in = out", $\sigma\, T_E^4 = (S_o/4)\,(1 - \alpha_p)$, we have

$$T_E = [(1/\sigma)(S_o/4)(1 - \alpha_p)]^{1/4}$$

For example, if $\alpha_p = 0.1$, we have T_E ⬆ 271.4 K

Note:

(1) Given T_E = 271.4 K, we can plug it into $B_\lambda(T)$ to see that the peak wavelength in this case is ~10 μm, in the infrared range.
==> Earth emits primarily infrared radiation

(2) The previous slide has not taken into account the fact that Earth's atmosphere can absorb infrared radiation (and re-emitting it back to the surface), preventing heat loss to space. The consequence is that Earth's surface would be warmer in the presence of "greenhouse gases" in the atmosphere

For our purpose, the "greenhouse gases" are defined as the gases that absorb the infrared radiation emitted by a planet

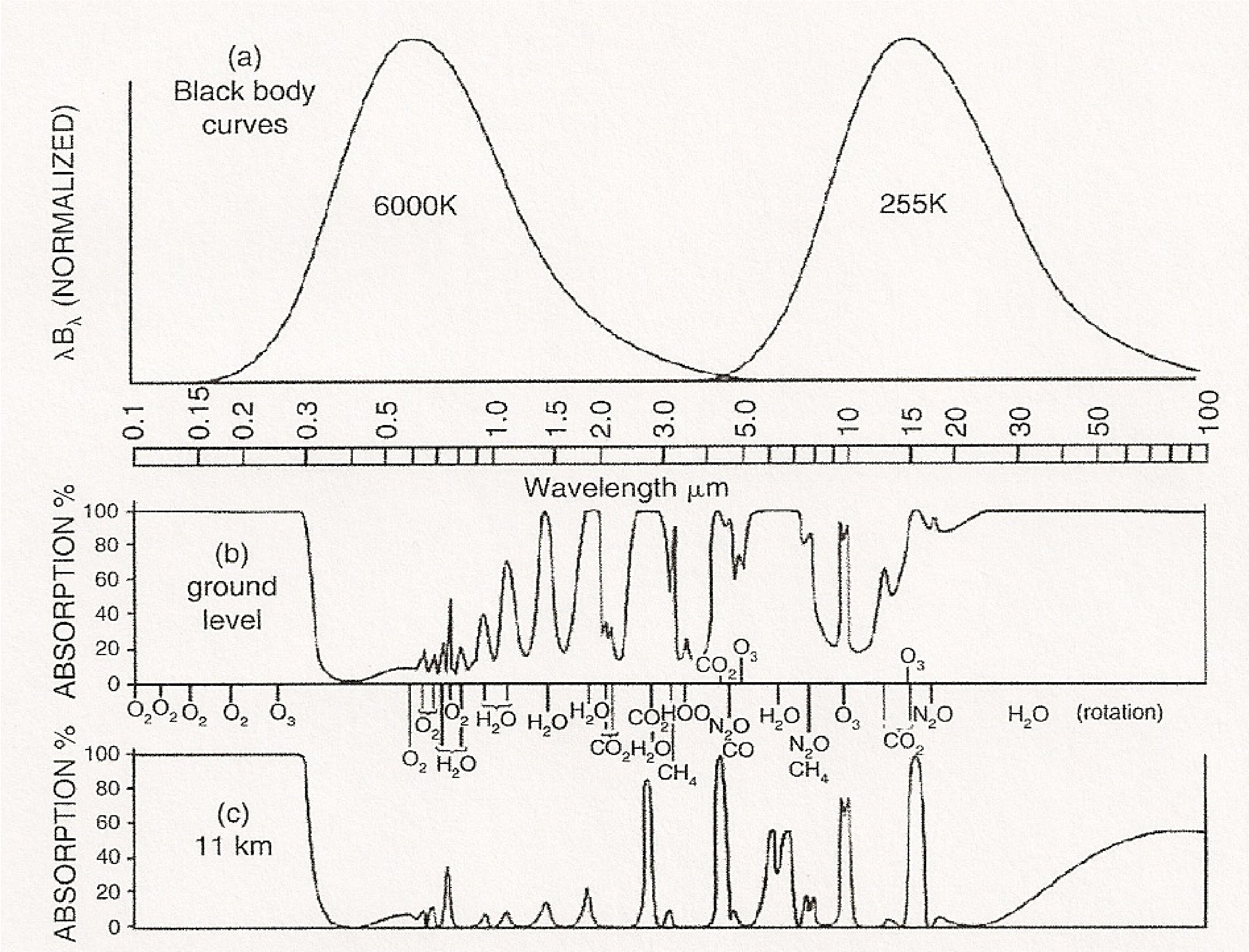
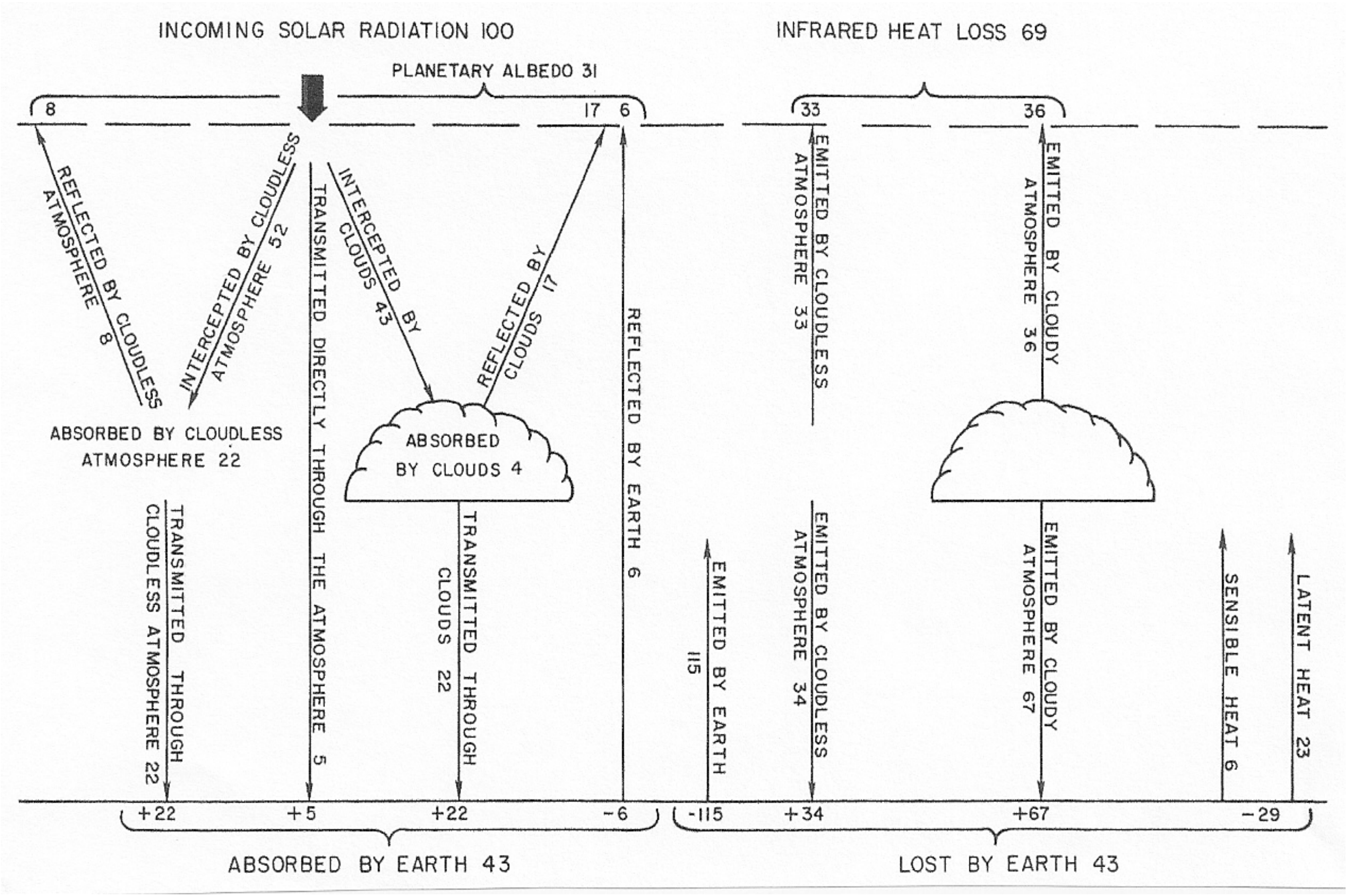


Fig. 2.6 in M&P (again)

Energy balance of global atmosphere (will be revisited)



From Liou (1980) (Not recent; might need some update)