MAE578, Spring 2019 Homework #4
See HW1,2 for rules on collaboration and guideline on submission of report

Prob 1. (35%) In the vicinity of the City of Mesa, marked by a circle in Fig. 1, a rectangular grid of meteorological stations was set up to collect observations of temperature, velocity, etc. The spacing of the stations is 50 km in both *x*- and *y*-direction. At 9 PM of a certain day, the observation of the flow at the level of $z = 3$ km indicates that (see illustration in Fig. 1) (i) The horizonal velocity is uniform in space and points to northeast with a speed of 3 m/s. The wind vectors (red arrows) form a 45° angle with the *x*-axis, and (ii) The temperature contours (bold gray lines) are straight and equally spaced lines that form a 30° angle with the *x*-axis. The magnitude of temperature gradient is as shown in Fig. 1. At the level of $z = 3$ km, the temperature at Mesa is 9°C. Moreover, from the observations at other vertical levels, it has been determined that (iii) The lapse rate of temperature is $\Gamma = -\partial T/\partial z = 6$ °C/km in the area of interest.

From the observations of the horizontal velocity at multiple vertical levels, your colleagues have performed a vertical integration of the horizontal wind divergence to determine that (iv) The vertical velocity at $z = 3$ km is approximately –5 cm/s (i.e., a subsidence), uniformly over the area of interest. Lastly, your colleagues have also performed a radiative transfer calculation to estimate that (v) The radiative heating rate $\dot{Q}/c_p$ of the atmosphere at z = 3 km is approximately –2 °C/hour (i.e., it is actually cooling over time), uniformly over the area of interest. Using the information from (i)-(v) and ignoring the effect of moisture, try to make a prediction of the temperature at 10 PM, at $z = 3$ km at the location of Mesa. You may use the the standard temperature equation in z-coordinate:

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} + (\Gamma - \Gamma_d)w + \dot{Q}/c_p, \qquad \text{Eq. (1)}$$

where $\Gamma_d \equiv g/c_p$ is the dry adiabatic lapse rate. [We expect this to be a simple exercise of integrating Eq. (1) numerically forward by one time step, with $\Delta t = 1$ hour.]

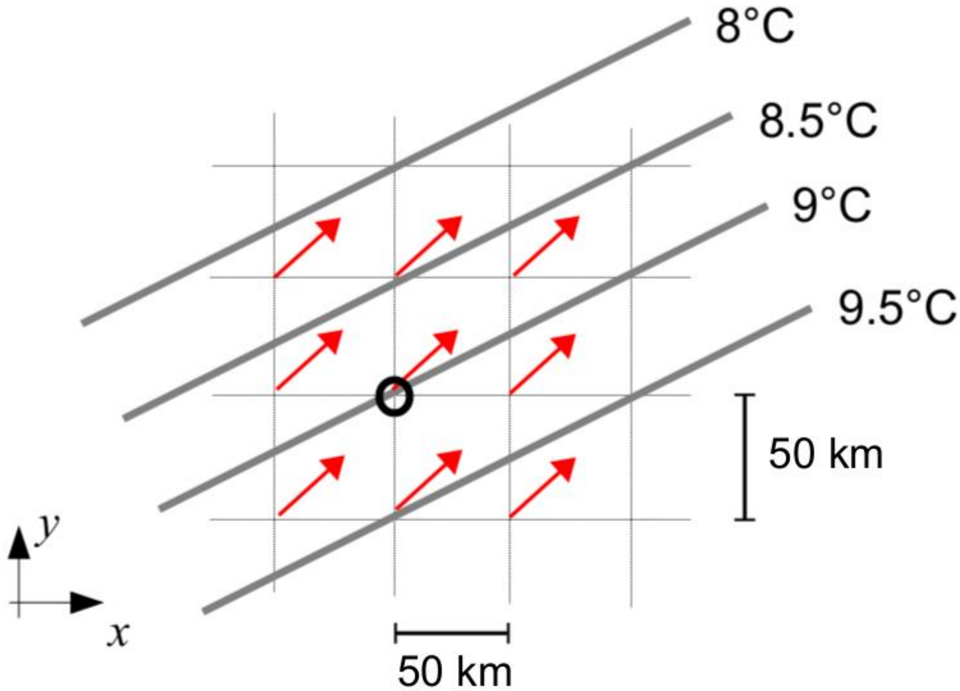

Fig. 1

Prob 2 (45%) Consider a system of an air flow blowing through a street canyon in an urban area, as illustrated in Fig. 2. The street canyon is parallel to the *y*-direction. Over a certain period of time, an approximately steady state is observed with the following characteristic scales for the flow:

- The horizontal length scales in *x*- and *y*-direction are $L_x \sim 10$ m and $L_y \sim 100$ m, respectively. The vertical length scale is $H \sim 10$ m.

- The scales for the velocity components in *x*- and *y*-direction are $U \sim 1$ m/s and $V \sim 10$ m/s, respectively. The scale of vertical velocity is $W \sim 0.1$ m/s.

(a) Assuming that density is constant, ignoring Coriolis effect but including the effect of gravity, the steady version of Navier-Stokes equations and continuity equation (all in z-coordinate) is given as (all notations are standard)

$$0 = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \text{Eq. (2)}$$

$$0 = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{Eq. (3)}$$

$$0 = -u\frac{\partial w}{\partial x} - v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - g \qquad \text{Eq. (4)}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad \text{Eq. (5)}$$

Perform a scale analysis for all terms in the above 4 equations. Clearly write out the scale for each term. (You may assume that pressure gradient force balances with the dominating term(s) among all other terms in the equation.) Then, reduce the four equations, Eq. (2)-(5), to a simplified version in which only the leading-order terms in each equation are retained.

(b) From the analysis in Part (a), determine the order of magnitude of the ratio, $|-\rho^{-1}\partial p/\partial y|/|-\rho^{-1}\partial p/\partial x|$. (This is the ratio of the "magnitude of PGF in *y*-direction" to "magnitude of PGF in *x*-direction".) Is the result consistent with the given observation that the velocity field is dominated by a strong flow in the *y*-direction?

(c) If Coriolis force is introduced to the horizontal components of the momentum equations, the first two equations in Part (a) would become

$$0 = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \text{Eq. (2A)}$$

$$0 = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{Eq. (3A)},$$

where $f = 2\,\Omega\sin(\varphi)$ is the Coriolis parameter, $\Omega = (2\pi)/(1\text{ day})$ is the rotation rate of the Earth, and φ is latitude (φ = 0 at the equator, φ = π/2 at the North Pole). Assume that the urban site for this system is located at 30°N. Repeat the scale analysis in Part (a) for these two equations. (Essentially, just add the scales for the Coriolis terms.) For this system, do the Coriolis terms contribute to the leading-order balance of horizontal momentum?

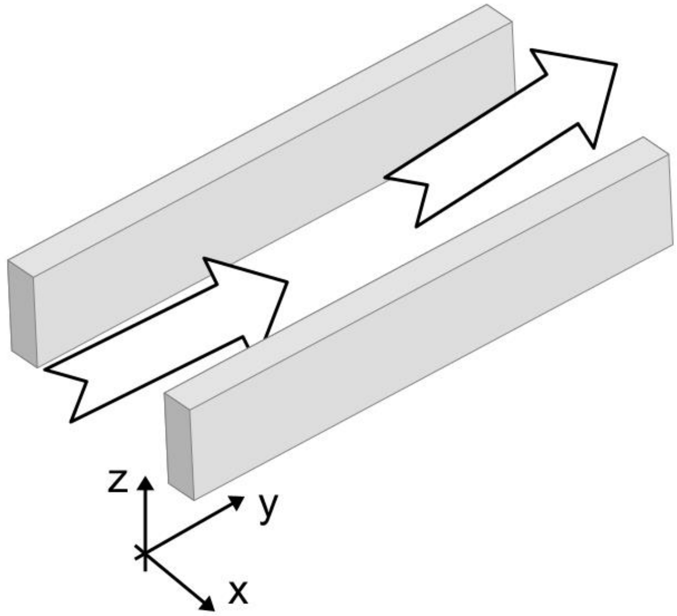

Fig. 2

Prob 3 (20%) If the atmosphere is approximately in hydrostatic balance in the vertical direction, pressure (p) can be used as an alternative vertical coordinate. We have discussed the benefit of writing the governing equations in p-coordinate. For example, the pressure gradient force (PGF) in x-direction under z-coordinate can be transformed into its counterpart in p-coordinate as

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p$$

(the l.h.s. is the PGF in z-coordinate, and r.h.s. the PGF in p-coordinate), where $\Phi = gz$ is geopotential. Density is invisible in the PGF in p-coordinate, because the information of vertical distribution of mass has been absorbed into the coordinate system. As mentioned in the lectures, if the atmosphere is statically stable, i.e., $\partial\theta/\partial z > 0$, one could possibly choose potential temperature (θ) as another alternative vertical coordinate. Show that the PGF in x-direction can be expressed in θ-coordinate as

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial E}{\partial x}\right)_\theta ,$$

where $E = C_pT + gz$ is the dry static energy. In other words, the horizontal component of PGF is equivalent to the horizontal gradient of dry static energy on an isentropic surface. (We only consider the x-component of PGF but the story for the y-component is the same.)