MAE578, Spring 2019 Homework #5
See HW1,2 for rules on collaboration and guideline on submission of report

Prob 1 (20%) Solve Problem 4 in Chapter 7 of the textbook. Plot the solution for the pressure and velocity fields. Since the system is axially symmetric, it suffices to make line plots of pressure and tangential velocity as a function of radius (r). An alternative is to make a 2-D contour plot for pressure (using the "contour" function in Matlab) and a plot of the 2-D vector field of horizontal velocity (using the "quiver" function in Matlab).

Prob 2 (20%) Solve Problem 8 in Chapter 7. To visualize the setup of the problem, you might find it useful to read the summary of observations in Chapter 5, particularly the discussions related to Figs. 5.7, 5.12, 5.13, 5.14, and 5.20.

Prob 3 (20%) Solve Problem 9 in Chapter 7. Note that "westerly" means "eastward". Also, assume that the system is in the Northern Hemisphere (so the "winter pole" in Fig. 7.29 is North Pole.)

Prob 4 (10%) At 45°N, a river runs from south to north with a uniform velocity of 5 cm/s. It is 1 km wide as illustrated in Fig. 1. Assume that the flow of the river is in geostrophic balance and ignore the effect of friction. Determine the difference in the depth of water across the river. You may assume that the density of water is $\rho = 1000\ \mathrm{kg/m^3}$. <u>In your answer, please clearly indicate whether the depth of water increases eastward or westward</u>.)

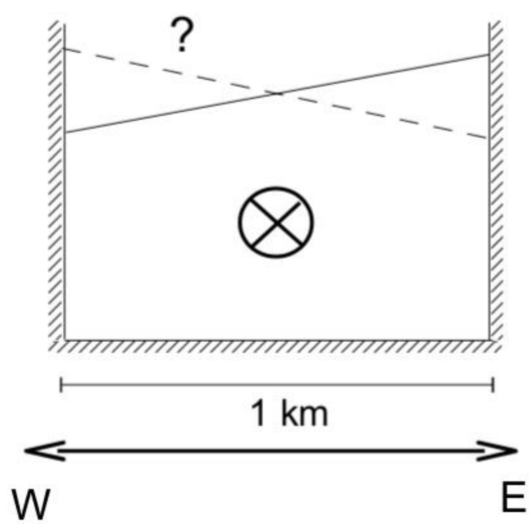

Fig. 1 *Schematic diagram for the setup of Prob 4. This is the cross-sectional view of the river from an observer facing north. (The river flows into the paper.)*

Prob 5 (30%) Ignoring viscosity, consider a 2-D flow in the rotating frame for which the momentum equations can be written as

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv \qquad (1)$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu \qquad (2),$$

where $f = 2\Omega \sin\varphi$ is the Coriolis parameter. For this problem, we assume that f is constant (justifiable over a small area), say is fixed at its value at 30°N.

(a) If density is uniform, show that Eqs. (1) and (2) lead to

$$\frac{d(\zeta+f)}{dt} = -(\zeta + f)D \qquad (3)$$

where $\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the *vorticity* and $D \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is the *divergence* of the 2-D velocity. [The d/dt in Eq. (3) is the total derivative, not to be confused with the partial derivative, $\partial/\partial t$, in Eqs. (1) and (2).] Therefore, if $D = 0$ we have conservation of $(\zeta + f)$ following the motion of a fluid parcel. The quantity, $\eta \equiv \zeta + f$, is the *absolute vorticity*. Here, we see that the Coriolis parameter is the "vorticity" associated to planetary rotation. In the literature, f is sometimes called the "planetary vorticity", and ζ the "relative vorticity".

[Note: To derive Eq. (3), it is not necessary to assume that f is constant. In general, f depends on latitude. Using a local Cartesian coordinate, one may approximate f as a linear function of y, $f = f_0 + \beta y$. One can readily show that Eq. (3) still holds with this non-constant f. For the exercise in Part (b), it is sufficient to consider f as a constant.]

(b) For a 2-D flow with constant density, by conservation of mass D has to be zero, unless there is a singularity of a mass source or sink. A situation that mimics the existence of such a mass source/sink is when one unplugs a large bathtub filled with water. The mass loss through the sinkhole would help maintain a negative 2-D (horizontal) divergence, $D < 0$. [In the context of Eq. (3), one can think of the "Lagrangian parcel" as a ring of fluid centered at the sinkhole. When $D < 0$, the parcel converges towards the sinkhole over time.] Assuming that D is a negative constant, solve Eq. (3) to show that the initial value of absolute vorticity would amplify (in amplitude, regardless of sign) over time. (For example, if η is initially negative, it will become even more negative over time.) Use this result to explain the following two phenomena:

(i) In daily life, if one unplugs a bathtub filled with water, a vortex tends to develop around the sinkhole. However, the vortex can rotate either clockwise or counterclockwise (i.e., the relative vorticity associated with the "mature" vortex can be either negative or positive) depending on the random detail of how one initially stirs the bathtub when pulling the plug. (And regardless of whether the experiment is done in the Northern or Southern Hemisphere.)

(ii) In contrast to (i), in "Perrot's experiment" described in Sec. 6.6.6 (p. 103) the vortex formed after one unplugs a large bathtub always rotates counterclockwise (i.e., $\zeta > 0$) if the experiment is performed in the Northern Hemisphere. Note that Perrot's experiment would work only if the water in the tank is left undisturbed for a long period of time before one pulls the plug from below. In other words, the tank of water has to be nearly motionless at $t = 0$.

[Hint: The key to explaining the difference between the two cases is to estimate the relative contribution of ζ and f to the initial value (therefore the sign) of η. In Case (i), when one dips a hand into a bathtub to unplug it, the initial disturbance created by the action has a length scale of ~ 10 cm (about the size of a hand) and a velocity scale of ~ 1 cm/s.]