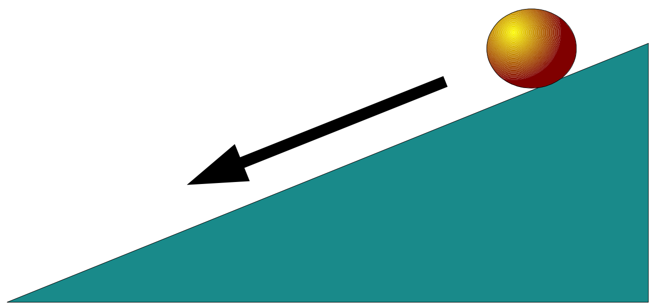
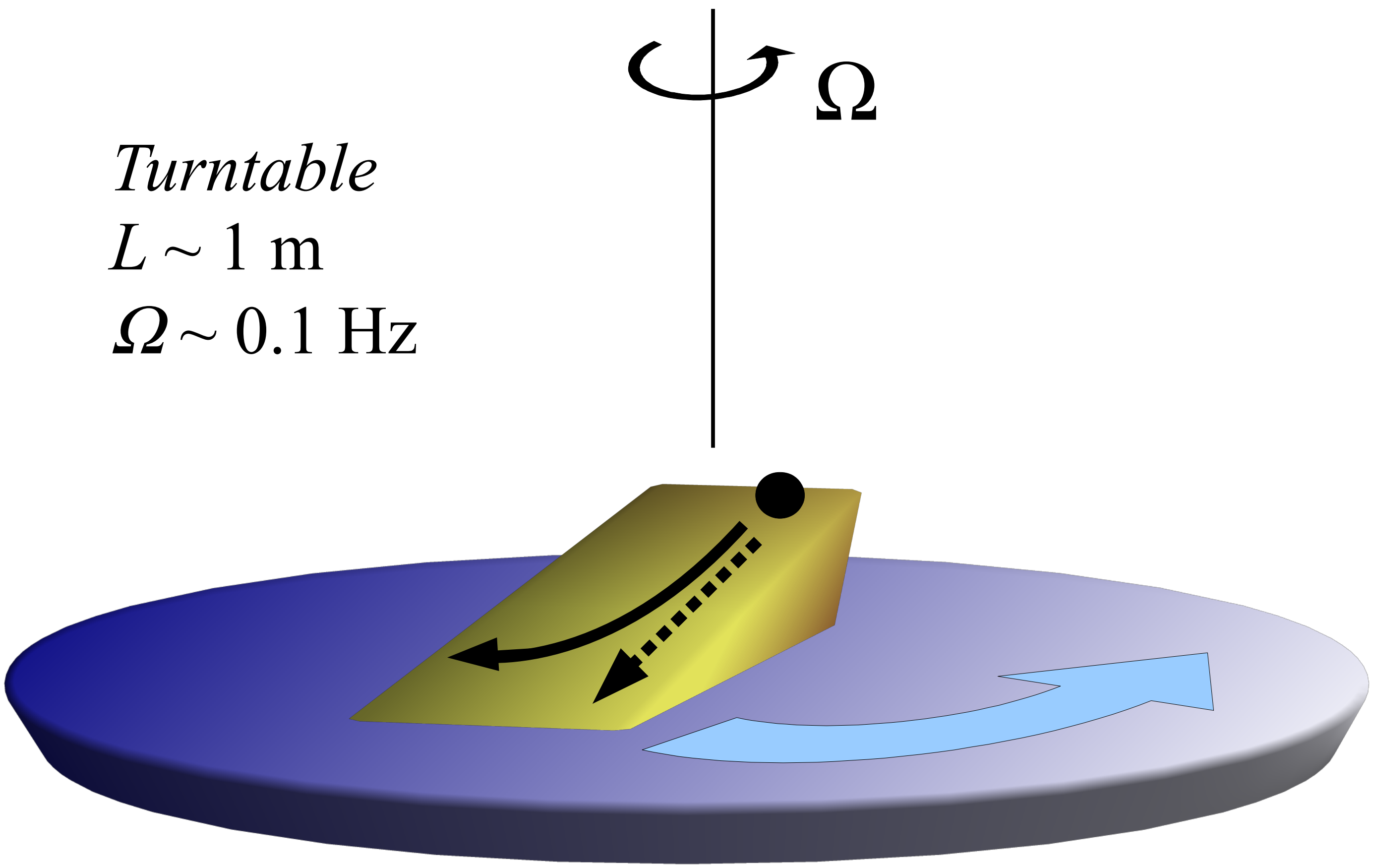
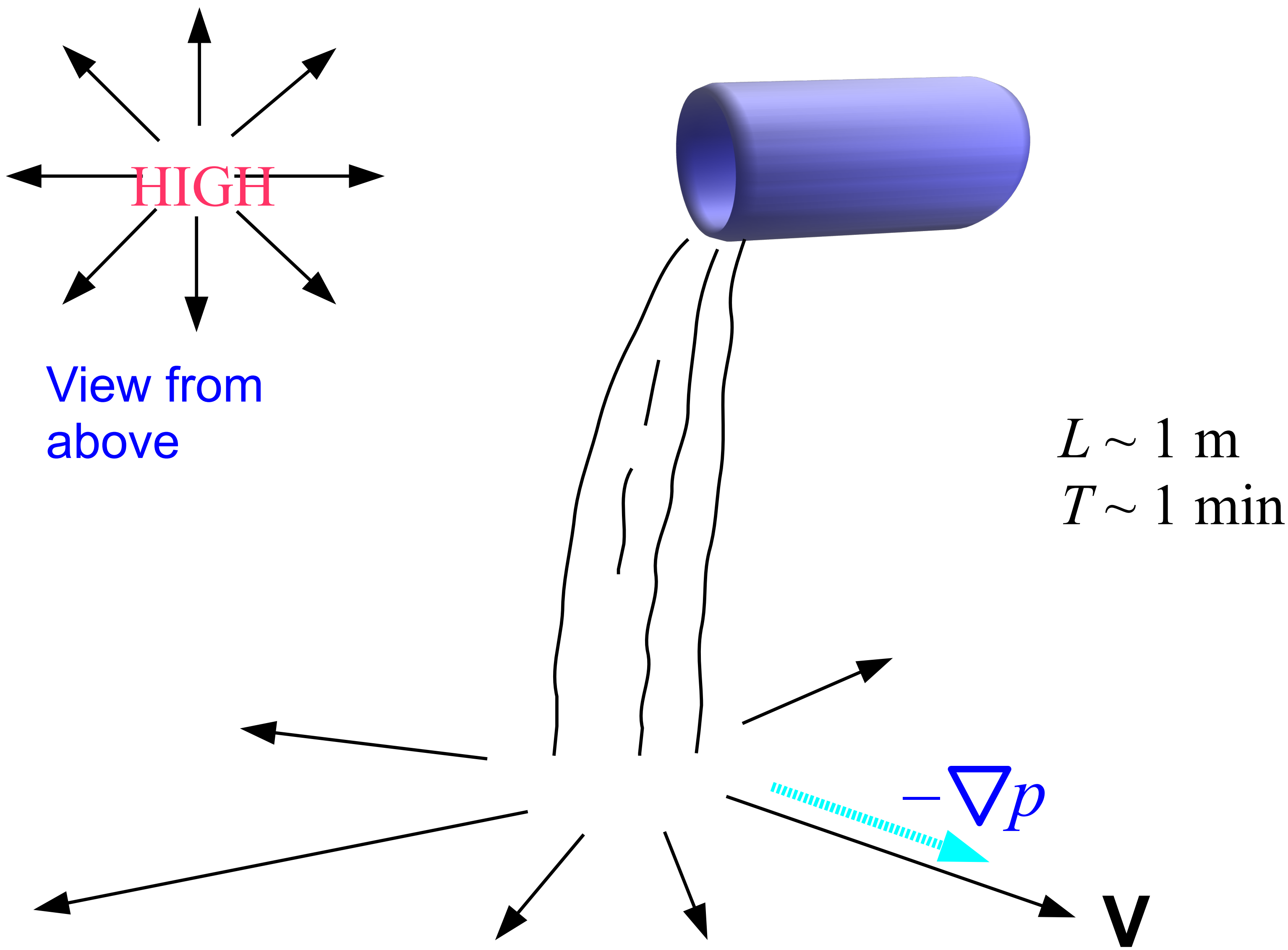
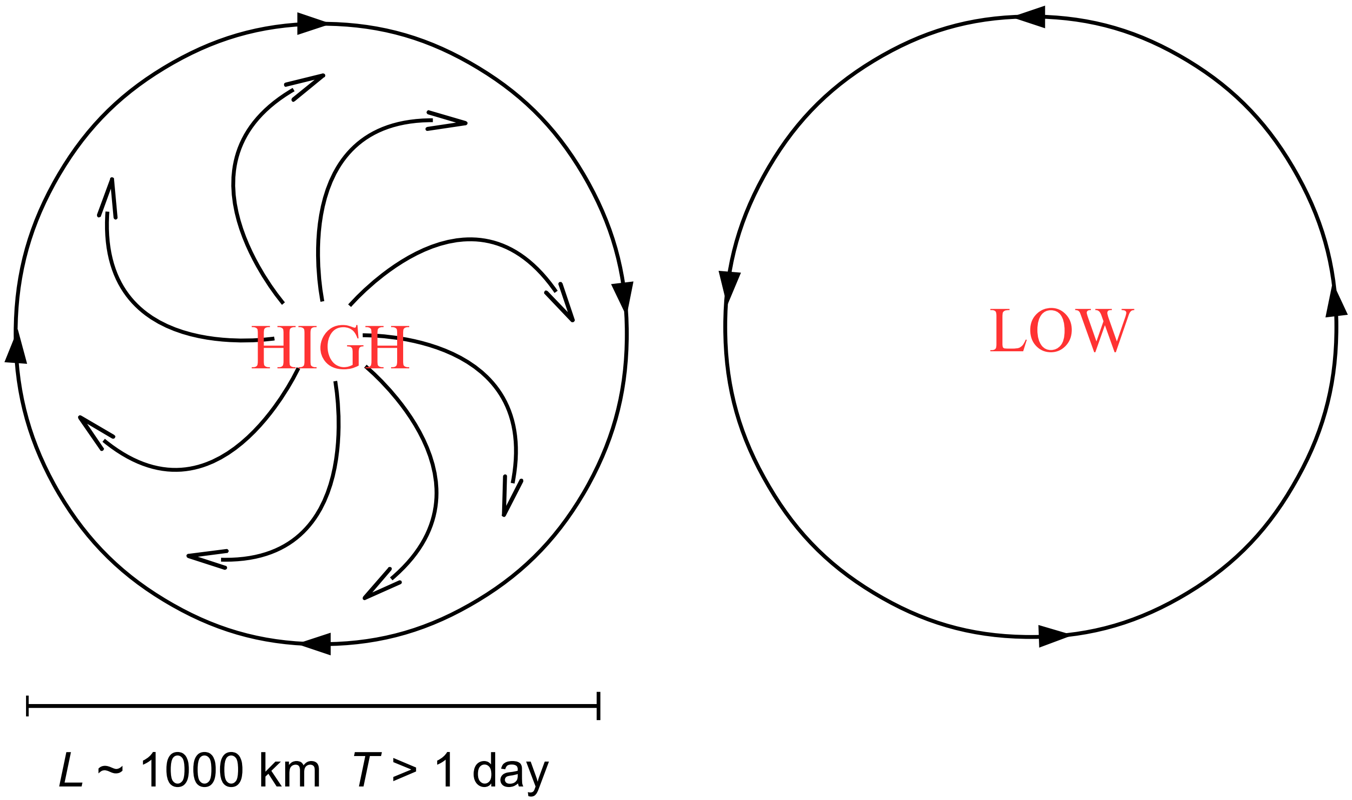
Coriolis effect





Earth rotation: $\Omega = 1$ cycle/day $= 0.000011$ Hz


HIGH
View from
above
$L \sim 1$ m
$T \sim 1$ min
$-\nabla p$
v



Large-scale flow :

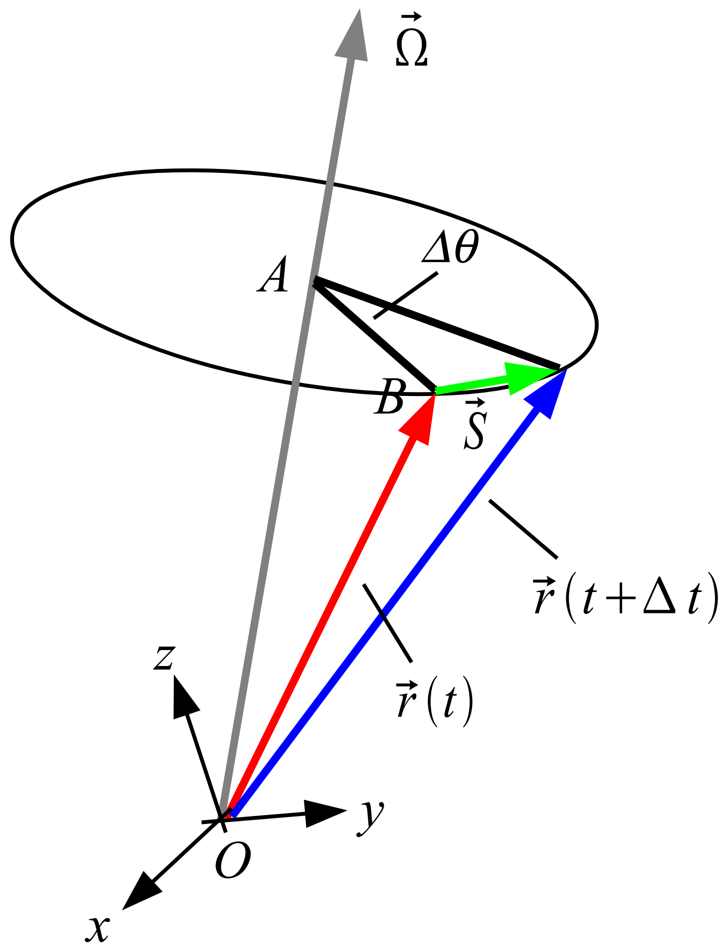
v *perpendicular to* ∇p

Rotation dominates (vorticity >> divergence) even if flow is not turbulent

We will see that the transformation of Navier-Stokes equations to a rotating frame is equivalent to adding a "Coriolis force" (and a "centrifugal force", which is however very small) to the momentum equation.

Navier-Stokes equations in a rotating coordinate system
Coriolis & centrifugal forces

Basic setup: Consider a vector, $\vec{r}$, that rotates with angular velocity Ω and with its axis of rotation pointing at the direction of $\hat{\Omega}$ (where $\hat{\Omega}$ is a unit vector)



$$\vec{S} \equiv \vec{r}(t+\Delta t) - \vec{r}(t)$$

$\vec{\Omega} \equiv \Omega\, \hat{\Omega}$ is the rotation vector

$$\Delta\theta = \Omega\, \Delta t$$

$|\vec{S}| = |\overline{AB}|\, \Delta\theta$ (as $\Delta\theta \rightarrow 0$)

$\longrightarrow \quad |\vec{S}| = |\overline{AB}|\, \Omega\, \Delta t$

$\vec{S}$ is normal to the plane spanned by $\vec{r}$ and $\vec{\Omega}$

$$\longrightarrow \quad \hat{S} = \frac{\vec{\Omega}\times\vec{r}(t)}{|\vec{\Omega}\times\vec{r}(t)|} \; ,$$

where $\hat{S}$ is the unit vector in $\vec{S}$ direction

$$\overline{OA} \perp \overline{AB} \;\longrightarrow\; |\overline{OA}| = \hat{\Omega}\cdot\vec{r}(t)$$

$$\longrightarrow \quad |\overline{AB}|^2 = |\overline{OB}|^2 - |\overline{OA}|^2 = |\vec{r}(t)|^2 - (\hat{\Omega}\cdot\vec{r}(t))^2 = |\hat{\Omega}|^2|\vec{r}(t)|^2 - (\hat{\Omega}\cdot\vec{r}(t))^2 = |\hat{\Omega}\times\vec{r}(t)|^2$$

$$\longrightarrow \quad |\overline{AB}| = |\hat{\Omega}\times\vec{r}(t)| \;\longrightarrow\; |\vec{S}| = |\hat{\Omega}\times\vec{r}(t)|\,\Omega\,\Delta t = |\vec{\Omega}\times\vec{r}(t)|\,\Delta t$$

(See Appendix A at end of slide set)

Therefore, $\vec{S} \equiv |\vec{S}|\hat{S} = \vec{\Omega}\times\vec{r}(t)\ \Delta t$, or $\dfrac{(\vec{r}(t+\Delta t) - \vec{r}(t))}{\Delta t} = \vec{\Omega}\times\vec{r}(t)$

As $\Delta t \to 0$, we have $\dfrac{d\vec{r}}{dt} = \vec{\Omega}\times\vec{r}$

This is the rate of change for the r-vector as measured in the inertial frame.
More precisely (the subscript "in" indicates "inertial" frame),

$$\left(\frac{d\vec{r}}{dt}\right)_{in} = \vec{\Omega}\times\vec{r} \ . \qquad (1)$$

An observer that rotates at the angular velocity and direction of $\vec{\Omega}$ will see the r-vector as steady, not moving at all, i.e., (the subscript "rot" indicates rotating frame),

$$\left(\frac{d\vec{r}}{dt}\right)_{rot} = 0 \ . \qquad (2)$$

Comparing (1) and (2), we see that the difference between $\left(\dfrac{d\vec{r}}{dt}\right)_{in}$ and $\left(\dfrac{d\vec{r}}{dt}\right)_{rot}$ is $\vec{\Omega}\times\vec{r}$,

$$\left(\frac{d\vec{r}}{dt}\right)_{in} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\Omega}\times\vec{r} \ . \qquad (3)$$

Note that the $\vec{r}$ here can be any vector.

1. Velocity

If $\vec{r}$ is the vector of the location of an air parcel, $\frac{d\vec{r}}{dt}$ would be its velocity so we have

$$\vec{v}_{in} = \vec{v}_{rot} + \vec{\Omega}\times\vec{r} \ .$$

2. Acceleration

Now, we need to apply the relation, $\left(\frac{d}{dt}\right)_{in} = \left[\left(\frac{d}{dt}\right)_{rot} + \vec{\Omega}\times\right]$, to $\vec{v}_{in}$:

$$\begin{aligned}\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} &= \left(\frac{d\vec{v}_{in}}{dt}\right)_{rot} + \vec{\Omega}\times\vec{v}_{in} \\ &= \left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} + \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times\vec{\Omega}\times\vec{r} \\ &= \left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times\vec{\Omega}\times\vec{r}\end{aligned}$$

3. Equation of motion

Since Newton's law is $\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \frac{\vec{F}}{m}$, in the rotating frame we have

$$\left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} = \frac{\vec{F}}{m} - 2\vec{\Omega}\times\vec{v}_{rot} - \vec{\Omega}\times\vec{\Omega}\times\vec{r}$$

Since $\frac{\vec{F}}{m} = -\frac{1}{\rho}\nabla p - g\,\hat{z}$ for a fluid parcel (ignoring molecular viscosity - recall the scale analysis for large-scale flows), the momentum equation for the fluid parcel under the rotating frame can now be written as

$$\left(\frac{d\vec{v}}{dt}\right) = -\frac{1}{\rho}\nabla p - g\,\hat{z} - \underbrace{2\vec{\Omega}\times\vec{v}}_{(A)} - \underbrace{\vec{\Omega}\times\vec{\Omega}\times\vec{r}}_{(B)} \ .$$

The momentum equation in the rotating coordinate system has two extra terms:

Term (A): Coriolis force   Term (B): Centrifugal force

Although those terms are sometimes called "fictitious forces" (that arise from a coordinate transformation), they are real in that they absorb or replace the effect of Earth rotation. They are perpendicular to the velocity vector so can only act to change the "direction" of motion but not the net kinetic energy of the flow.

Expanding $\left(\frac{d\vec{v}}{dt}\right)$ into its Eulerian form, we have Navier-Stokes equations in the rotating frame:

$$\frac{\partial\vec{v}}{\partial t} = -\,\vec{v}\cdot\nabla\vec{v} - \frac{1}{\rho}\nabla p - g\,\hat{z} - \underbrace{2\vec{\Omega}\times\vec{v}}_{(A)} - \underbrace{\vec{\Omega}\times\vec{\Omega}\times\vec{r}}_{(B)} \ .$$

Appendix A

We will show that $|\vec{A}|^2|\vec{B}|^2 - (\vec{A}\cdot\vec{B})^2 = |\vec{A}\times\vec{B}|^2$.

By definition, $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$ vectors, while $(\vec{A}\cdot\vec{B}) = |\vec{A}||\vec{B}|\cos\theta$. Thus, we have

$$|\vec{A}\times\vec{B}|^2 + (\vec{A}\cdot\vec{B})^2 = |\vec{A}|^2|\vec{B}|^2(\cos^2\theta+\sin^2\theta) = |\vec{A}|^2|\vec{B}|^2 .$$