Governing equations of fluid dynamics under the influence of Earth rotation (Navier-Stokes Equations in rotating frame)

Recap:

From kinematic consideration,

$$\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times\vec{\Omega}\times\vec{r} \ , \qquad (1)$$

where "in" and "rot" denote the inertial (absolute) and rotating frames.

Equation of motion

Since Newton's law is $\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \frac{\vec{F}}{m}$ in the inertial frame, in the rotating frame we have

$$\left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} = \frac{\vec{F}}{m} - 2\vec{\Omega}\times\vec{v}_{rot} - \vec{\Omega}\times\vec{\Omega}\times\vec{r} \ . \qquad (2)$$

Ignoring friction (viscosity), $\frac{\vec{F}}{m} = -\frac{1}{\rho}\nabla p - g\hat{z}$ as discussed before. The momentum equation for an air parcel in the rotating frame can now be written as

$$\left(\frac{d\vec{v}}{dt}\right) = -\frac{1}{\rho}\nabla p - g\hat{z} - \underbrace{2\vec{\Omega}\times}_{(A)}\vec{v} - \underbrace{\vec{\Omega}\times\vec{\Omega}\times}_{(B)}\vec{r} \ . \qquad (3)$$

The momentum equation in the rotating coordinate system has two extra terms:

Term (A): Coriolis force   Term (B): Centrifugal force

They are "fictitious forces" that arises from the coordinate transformation. They are perpendicular to the velocity vector so can only act to change the "direction" of motion but not the net kinetic energy of the flow.

Expanding $(\frac{d\vec{v}}{dt})$ into its Eulerian form, we have Navier-Stokes equations in the rotating frame:

$$\frac{\partial \vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \frac{1}{\rho}\nabla p - g\hat{z} - 2\vec{\Omega}\times\vec{v} - \vec{\Omega}\times\vec{\Omega}\times\vec{r} \quad . \tag{4}$$

Note that the centrifugal force is always perpendicular to the axis of rotation and is pointing "outward". Coriolis force is always perpendicular to the velocity vector. In the Northern Hemisphere, it points to the right of the velocity vector. For example, an eastward motion will be deflected southward.

In practice, we often combine $-g\hat{z} - \vec{\Omega}\times\vec{\Omega}\times\vec{r}$ into an effective gravity term (cf. Sec 6.6.5), $-\tilde{g}\hat{\tilde{z}}$. Here, $-g\hat{z}$ is the true gravity and points away from the center of the Earth. The vector, $\hat{\tilde{z}}$, on the other hand, is the true "local vertical" (and is the upward-pointing unit vector ***k*** that we will use later). We have discussed this point in detail in class, which is not repeated here.

Hereafter, let's use the effective gravity in our equations:

$$\frac{\partial \vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \frac{1}{\rho}\nabla p - g\boldsymbol{k} - 2\vec{\Omega}\times\vec{v} \ . \qquad (5)$$

Let (x, y, z) be the local Cartesian coordinate (see Fig. 6.19) and (***i***, ***j***, ***k***) the unit vectors in the x, y, and z directions. Write the rotation vector in its components for the local coordinate,

$$\vec{\Omega} \equiv (\Omega_x, \Omega_y, \Omega_z) \ ,$$

we have (again, consult Fig. 6.19)

$$\Omega_x = \vec{\Omega}\cdot\boldsymbol{i} = 0 \ , \quad \Omega_y = \vec{\Omega}\cdot\boldsymbol{j} = \Omega\cos(\phi) \ , \text{ and } \quad \Omega_z = \vec{\Omega}\cdot\boldsymbol{k} = \Omega\sin(\phi) \ ,$$

where ϕ is latitude and $\Omega \equiv |\vec{\Omega}|$ is the magnitude of the rotation vector, or simply the "rotation rate" of the planet. For the Earth, $\boldsymbol{\Omega}$ = 2π/(1 Earth day) = 2π/(86400 s) ≈ 7.27×10^{-5} s^{-1}. Next, the Coriolis force can be written in its three components in the local coordinate as

$$-2\vec{\Omega}\times\vec{v} = -2\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix} = -2(\Omega_y w - \Omega_z v)\boldsymbol{i} - 2(\Omega_z u - \Omega_x w)\boldsymbol{j} - 2(\Omega_x v - \Omega_y u)\boldsymbol{k}$$

$$= -2(w\,\Omega\cos\phi - v\,\Omega\sin\phi)\boldsymbol{i} - 2u\,\Omega\sin\phi\,\boldsymbol{j} + 2u\,\Omega\cos\phi\,\boldsymbol{k}$$

Lagrangian description

The momentum equations for an air parcel are

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2w\Omega\cos\phi + 2v\Omega\sin\phi$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2u\Omega\sin\phi$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2u\Omega\cos\phi$$

Eulerian description

The momentum equations for the velocity field are

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} - 2w\Omega\cos\phi + 2v\Omega\sin\phi$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2u\Omega\sin\phi$$

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2u\Omega\cos\phi$$

Some simplified cases

(We will later solidify the discussion with a detailed scale analysis using the Eulerian form of the equations)

1. For large-scale atmospheric motion, horizontal velocity is usually much greater than vertical velocity, $|w\,\Omega\cos\phi| \ll |v\,\Omega\sin\phi|$, while the horizontal pressure gradient force is comparable in magnitude to the horizontal component of Coriolis force. (All of the "horizontal" and "vertical" here refers to the local Cartesian coordinate.)

2. In the vertical direction, hydrostatic balance holds to a high degree of accuracy

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2v\,\Omega\sin\phi$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2u\,\Omega\sin\phi$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Later, we will show that du/dt and dv/dt are usually smaller than the horizontal pressure gradient and Coriolis terms. Let's however keep them for the current discussion.

Coriolis parameter

We often define $f \equiv 2\Omega \sin\phi$ as the Coriolis parameter. The horizontal momentum equations in preceding page can then be written as

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

Note that f is not a constant. It decreases toward the equator.

1. Inertial oscillation

If (1) there's no horizontal pressure gradient, and (2) the meridional (north-south) excursion of fluid motion is small enough for us to consider $f \equiv 2\Omega\sin\phi \approx 2\Omega\sin\phi_0 \equiv f_0$ (ϕ_0 is a reference latitude), then the equations of (horizontal) motion of an air parcel become

$$\frac{du}{dt} = f_0 v \ ,$$

$$\frac{dv}{dt} = -f_0 u \ .$$

The two equations can be combined into

$$\frac{d^2u}{dt^2} = -f_0^2 u \quad \Rightarrow \quad u(t) = u(0)\exp(i f_0 t) \quad \text{(solution for } v(t) \text{ is similar)} ,$$

which describes an oscillation with frequency f_0, or period $2\pi/f_0 = \pi/(\Omega \sin\phi_0)$, which is typically on the order of 1 day in midlatitude.

2. Geostrophic balance

For steady motion (no acceleration), du/dt = 0, dv/dt = 0, the horizontal velocity is related to pressure field by (we have neglected the Coriolis force related to *w*)

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f\,v$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - f\,u \quad .$$

This is called "geostrophic balance". It may be written in vector form as

$$f\,\boldsymbol{k}\times\vec{v} = -\frac{1}{\rho}\nabla\,p$$, where ***k*** is the local vertical normal vector (pointing upward)

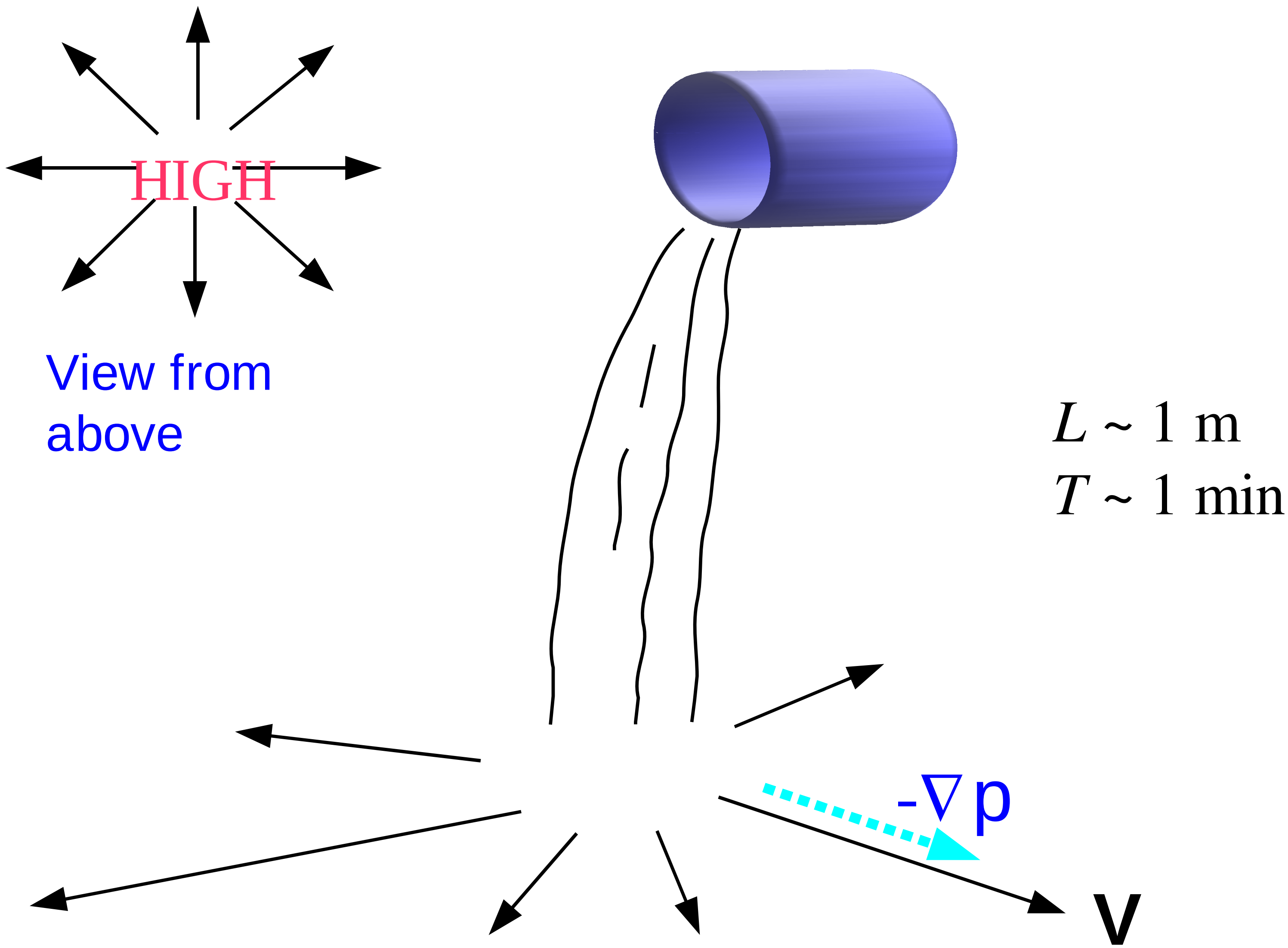
or,

$$\vec{v} = \frac{1}{f\,\rho}\boldsymbol{k}\times\nabla\,p \quad .$$

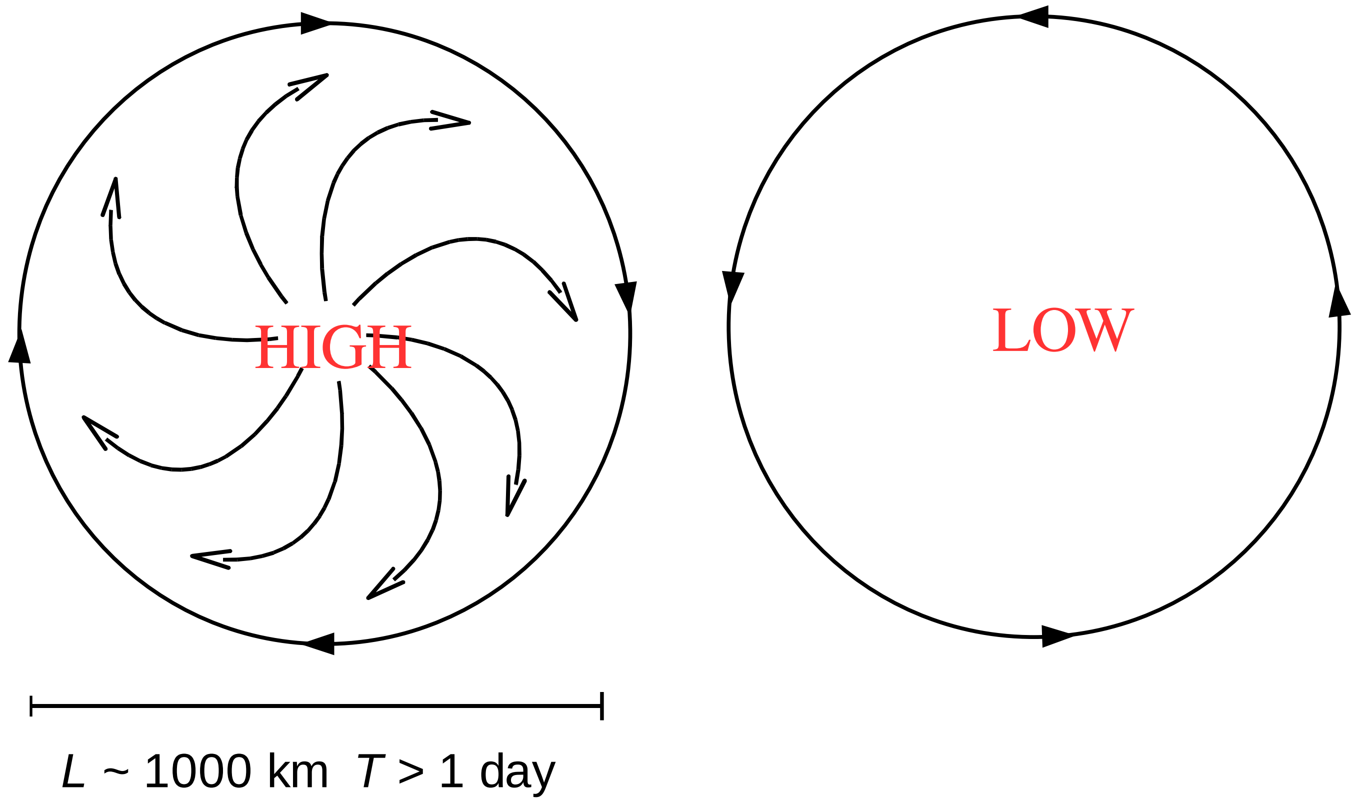
In geostrophic balance, the velocity vector is perpendicular to pressure gradient vector (or parallel to pressure contours). This is in dramatic contrast to small-scale motion, for which the velocity vector is often parallel to pressure gradient vector.

Recall our illustration of the contrast between small-scale and large-scale motion

Small-scale flow:



Large-scale flow :



$\mathbf{v}$ *perpendicular to* ∇p

Rotation dominates (vorticity >> divergence) even if flow is not turbulent