MAE598 Fall 2012 HW5
The difficulty level of this exercise is "**M**".

Consider the linear advection equation,

$$\frac{\partial u}{\partial t} = -C\frac{\partial u}{\partial x} \quad , \qquad (1)$$

where C is a positive constant. Using the 1st order forward scheme for the differentiation in t and the second-order central difference scheme for the differentiation in x, the finite difference form of Eq. (1) can be derived as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -C\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \quad , \qquad (2)$$

where $u_j^n \equiv u(j\Delta x, n\Delta t)$.

(a) Define the energy of the system as

$$E(n) \equiv \sum_j (u_j^n)^2 \quad ,$$

use the "Energy method" discussed in class to show that the finite difference scheme in Eq. (2) is unconditionally unstable, i.e., $E(n+1)/E(n) > 1$ regardless of how small $C\Delta t/\Delta x$ is. Note that for the analytic solution we would have $E(n+1)/E(n) = 1$.

(b) Perform numerical integration of Eq. (2) using the simple initial condition with a single "spike",

$$u(j\Delta x, 0) = \begin{cases} 1 & \text{, if } j = J \\ 0 & \text{, otherwise.} \end{cases}$$

Consider the three cases with $\mu = 0.1$, 0.5, and 0.9 (where $\mu \equiv C\Delta t/\Delta x$). In each case, integrate the system for 10 time steps. Plot $E(n)$ as a function of n (for $n = 0, 1, 2, ..., 10$) for the three cases. The outcome should corroborate your finding in (a), that the energy grows in time for the numerical solution.

(c) Use von Neuman's method to show that the finite difference scheme in Eq. (2) is also unconditionally unstable in the sense of von Neumann.