Prob 2(b)

Sep. of var. $u \sim G(x)H(y) \Rightarrow C = \frac{\ddot{H}}{H} = -\frac{G''}{G}$

$(\dot{\ }) \equiv \frac{d()}{dy}$

$(\ )' \equiv \frac{d()}{dx}$

b.c. (iii), (iv) $\rightarrow$ $\dot{H}(0)=0 \quad \dot{H}(1)=0$

$C = 0, -\pi^2, -4\pi^2, \ldots \quad H_0(y) = 1, \quad H_n(y) = \cos(n\pi y)$
$n = 1, 2, 3, \ldots$

$G_n'' = (n\pi)^2 G_n \Rightarrow G_0(x) = A_0 x + B_0$

$G_n(x) = A_n \cosh(n\pi x) + B_n \sinh(n\pi x), \quad n = 1, 2, \ldots$

From b.c. (i) & (ii), only $n=0$ and $n=2$ matter.

$\Rightarrow$ Full solution $u(x,y) = A_0 x + B_0 + A_2 \cosh(2\pi x)\cos(2\pi y) + B_2 \sinh(2\pi x)\cos(2\pi y)$.

$\rightarrow u_x(x,y) = A_0 + 2\pi A_2 \sinh(2\pi x)\cos(2\pi y) + 2\pi B_2 \cosh(2\pi x)\cos(2\pi y)$.

From b.c. (i): $A_0 + 2\pi B_2 \cos(2\pi y) = 3 + \cos(2\pi y)$

$\rightarrow A_0 = 3, \quad B_2 = \frac{1}{2\pi}$

From b.c. (ii): $A_0 + 2\pi[A_2 \sinh(2\pi) + B_2 \cosh(2\pi)]\cos(2\pi y) = 3$

$\rightarrow A_0 = 3, \quad A_2 \sinh(2\pi) + B_2 \cosh(2\pi) = 0$

$\Rightarrow A_2 = -\frac{\coth(2\pi)}{2\pi}$

Also, $B_0$ cannot be determined.

$\Rightarrow \quad u(x,y) = 3x + B_0 + \frac{1}{2\pi}[\sinh(2\pi x) - \coth(2\pi)\cosh(2\pi x)]\cos(2\pi y)$

$\hookrightarrow$ Multiple solutions, since $B_0$ can be of any value. #

---

Note: With the aid of trig identities for hyperbolic functions, the solution can also be expressed as

$$u(x,y) = 3x + B_0 - \frac{\cosh[2\pi(1-x)]\cos(2\pi y)}{2\pi\sinh(2\pi)}$$ #

Prob 3

Sep. of var. : $u \sim G(x)H(y) \rightarrow x^2 G''H + G\ddot{H} - 2x G'H = 0$

$$\rightarrow -\frac{(x^2G'' - 2xG')}{G} = \frac{\ddot{H}}{H} = C \qquad \left.\begin{matrix} H(0)=0 \\ H(\pi)=0 \end{matrix}\right\} \text{ b.c. (iii) \& (iv)}$$

$$C_n = -n^2, \quad H_n(y) = \sin(ny), \quad n = 1, 2, 3, \ldots$$

From b.c. (i) & (ii),
only $\underline{n=2}$ is relevant.
$(C_2 = -2^2 = -4)$

$\Rightarrow$ Suffices to solve $G_2(x)$ :

$$-\frac{(x^2G_2'' - 2xG_2')}{G_2} = C_2 = -4$$

$$\rightarrow x^2G_2'' - 2xG_2' - 4G_2 = 0$$

let $G_2(x) \sim x^p \Rightarrow p^2 - 3p - 4 = 0 \Rightarrow p = 4, -1$

$$\rightarrow G_2(x) = Ax^4 + Bx^{-1}$$

Full solution is $u(x,y) = G_2(x)H_2(y) = \left(Ax^4 + Bx^{-1}\right)\sin(2y)$

From b.c. (i) : $(A+B)\sin(2y) = 0 \Rightarrow A+B=0, \ \underline{B=-A}$

From b.c. (ii) : $(A\cdot 2^4 + B\,2^{-1})\sin(2y) = 31\sin(2y)$

$$\Rightarrow 16A + \frac{B}{2} = 31 \Rightarrow 16A - \frac{A}{2} = 31$$

$$\Rightarrow A = 2, \ B = -2$$

$$\rightarrow u(x,y) = 2\left(x^4 - \frac{1}{x}\right)\sin(2y)$$

#

Prob 4

Sep. of var.: $u \sim G(x)H(y) \rightarrow HG'' - 3HG' + G\ddot{H} = 0$

$$\rightarrow \quad -\frac{(G''-3G')}{G} = \frac{\ddot{H}}{H} = c \qquad \left.\begin{matrix}\dot{H}(0)=0\\ \dot{H}(\pi)=0\end{matrix}\right\} \text{b.c. (iii), (iv)}$$

$$\Downarrow$$

$$c = 0, \{-n^2, n=1,2,3\ldots\}$$

$$H_0(y) = 1, \quad H_n(y) = \cos(ny) \quad n=1,2,3,\ldots$$

From b.c. (i) & (ii), only $\underline{n=0}$ and $\underline{n=2}$ matter

$$\begin{matrix} n=0 & & n=2 \\ \downarrow & & \downarrow \\ c=0 & & c=-4 \end{matrix}$$

$n=0$: $\quad G_0'' - 3G_0' = 0 \quad$ let $G_0 \sim e^{px} \Rightarrow p^2 - 3p = 0$

$$p = \underline{3}, \underline{0}$$

$$\rightarrow G_0(x) = A_0 e^{3x} + B_0 \qquad (e^{0x} = 1)$$

$n=2$: $\quad G_2'' - 3G_2' - 4G_2 = 0$

let $G_2 \sim e^{px} \Rightarrow p^2 - 3p - 4 = 0$

$$p = 4, -1$$

$$\rightarrow G_2(x) = A_2 e^{4x} + B_2 e^{-x}$$

Full solution: $u(x,y) = (A_0 e^{3x} + B_0) + (A_2 e^{4x} + B_2 e^{-x})\cos(2y)$

From b.c. (i): $\quad (A_0 + B_0) + (A_2 + B_2)\cos(2y) = 1$

$$\Rightarrow A_0 + B_0 = 1, \qquad A_2 + B_2 = 0 \Longrightarrow B_2 = -A_2$$

From b.c. (ii): $\quad (A_0 e^3 + B_0) + (A_2 e^4 + B_2 e^{-1})\cos(2y) = e^3 + \cos(2y)$

$$\Rightarrow A_0 e^3 + B_0 = e^3, \qquad A_2 e^4 + B_2 e^{-1} = 1$$

$$\Rightarrow A_0 = 1, B_0 = 0 \qquad \Rightarrow A_2 = 1/(e^4 - e^{-1})$$

Full solution:

$$u(x,y) = e^{3x} + \left(\frac{e^{4x} - e^{-x}}{e^4 - e^{-1}}\right)\cos(2y) \qquad \#$$