Prob 4

F.S. expansion: $u(x,t) = \sum_n C_n(t)\, e^{inx}$

$$\Rightarrow \quad \ddot{C}_n = (-8n^2 + n^4 + 16)\, C_n$$

From the b.c.'s, only $n = 1, 2$ need to be retained

$$\ddot{C}_1 = 9\, C_1 \quad \Rightarrow \quad C_1(t) = A_1 \cosh(3t) + B_1 \sinh(3t)$$

$$\ddot{C}_2 = 0 \quad \Rightarrow \quad C_2(t) = A_2 t + B_2$$

$$\Rightarrow \quad u(x,t) = \left\{ [A_1 \cosh(3t) + B_1 \sinh(3t)]\, e^{ix} + (A_2 t + B_2)\, e^{i2x} \right\} + c.c.$$

From b.c. (i),

$$u(x,0) = \sin(2x) = \frac{1}{2i} e^{i2x} + c.c.$$

From b.c. (ii),

$$u_t(x,0) = \cos(x) = \frac{1}{2} e^{ix} + c.c.$$

$$\Rightarrow \quad u(x,0) = \left\{ A_1 e^{ix} + B_2 e^{i2x} \right\} + c.c.$$

$$u_t(x,0) = \left\{ 3B_1 e^{ix} + A_2 e^{i2x} \right\} + c.c.$$

compare

$$\Downarrow$$

$$A_1 = 0, \quad B_2 = \frac{1}{2i}, \quad B_1 = \frac{1}{6}, \quad A_2 = 0$$

$$\Rightarrow \quad u(x,t) = \left\{ \frac{1}{6} \sinh(3t)\, e^{ix} + \frac{1}{2i} e^{i2x} \right\} + c.c.$$

$$= 2\,\mathrm{Re} \underbrace{\left\{ \frac{1}{6} \sinh(3t)\, e^{ix} + \frac{1}{2i} e^{i2x} \right\}}_{= \left(\frac{1}{6} \sinh(3t) \cos(x) + \frac{1}{2} \sin(2x)\right) + (\text{imaginary part})}$$

$$= \frac{1}{3} \sinh(3t) \cos(x) + \sin(2x)$$

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Prob 5

F.S. expansion : $u(x,t) = \sum_n C_n(t)\, e^{inx}$

$$\Rightarrow \quad \dot{C}_n = \left[(1+2t)(-n^2) + t\,(-in^3) + n^4 + 2t\right] C_n$$

From the b.c.'s, only $n=0,1$ will survive.

$$\dot{C}_0 = 2t\,C_0 \quad \Rightarrow \quad \frac{dC_0}{C_0} = 2t\,dt \quad \Rightarrow \quad C_0(t) = C_0(0)\, e^{t^2}$$

$$\dot{C}_1 = -it\,C_1 \quad \Rightarrow \quad \frac{dC_1}{C_1} = -it\,dt \Rightarrow C_1(t) = C_1(0)\, e^{-it^2/2}$$

$$u(x,t) = C_0(0)\, e^{t^2} + \left\{ C_1(0)\, e^{-it^2/2} e^{ix} + c.c. \right\}$$

$$\Rightarrow \quad u(x,0) = C_0(0) + \left\{ C_1(0)\, e^{ix} + c.c. \right\}$$

From b.c. :

$$u(x,0) = 1 + \sin(x)$$

$$= 1 + \left\{ \frac{1}{2i} e^{ix} + c.c. \right\}$$

compare

$$\Downarrow$$

$$C_0(0) = 1, \quad C_1(0) = \frac{1}{2i}$$

$$\Rightarrow \quad u(x,t) = e^{t^2} + \Big\{ \underbrace{\frac{1}{2i} e^{-it^2/2} e^{ix}}_{=\left(\frac{-i}{2}\right)\left[\cos\left(x - \frac{t^2}{2}\right) + i \sin\left(x - \frac{t^2}{2}\right)\right]} + c.c. \Big\}$$

$$= \frac{1}{2}\sin\left(x - \frac{t^2}{2}\right) + (\text{imaginary part})$$

$$\Rightarrow \quad u(x,t) = e^{t^2} + \sin\left(x - \frac{t^2}{2}\right)$$

#

Prob 6

F.S. expansion: $u(x,t) = \sum_n C_n(t) e^{inx}$

$$\Rightarrow \quad \ddot{C}_n = -n^2 C_n + in\dot{C}_n + C_n$$

From b.c. (i) & (ii), only $n=0, 1$ will survive

$$\ddot{C}_0 = C_0 \quad \Rightarrow \quad C_0(t) = A_0 \cosh(t) + B_0 \sinh(t)$$

$$\ddot{C}_1 = -C_1 + i\dot{C}_1 + C_1 = i\dot{C}_1 \quad \text{— } (*)$$

To solve $(*)$, let $D \equiv \dot{C}_1 \Rightarrow \dot{D} = iD \Rightarrow D(t) = D(0)e^{it} = \dot{C}_1(0)e^{it}$

$$\Rightarrow \quad \dot{C}_1 = D = \dot{C}_1(0)e^{it}$$

$$\Rightarrow \quad C_1(t) = C_1(0) + \dot{C}_1(0)\int_0^t e^{it}dt$$

$$= C_1(0) + \dot{C}_1(0)\left(\frac{1}{i}\right)(e^{it}-1)$$

From b.c. (i): $u(x,0) = 1 \Rightarrow C_0(0) = 1, \ C_1(0) = 0$

b.c. (ii): $u_t(x,0) = \cos(x) = \frac{1}{2}e^{ix} + \text{c.c.} \Rightarrow \dot{C}_0(0) = 0, \ \dot{C}_1(0) = \frac{1}{2}$

$$\Rightarrow \quad \cancel{C_0(t) =} \ A_0 = 1, \ B_0 = 0 \ \Rightarrow \ C_0(t) = \cosh(t)$$

$$C_1(t) = \frac{1}{2}\left(\frac{1}{i}\right)(e^{it}-1)$$

$$u(x,t) = C_0(t) + [C_1(t)e^{ix} + \text{c.c.}]$$

$$= \cosh(t) + \left\{\underbrace{\frac{1}{2i}(e^{it}-1)e^{ix}}_{= \frac{-i}{2}e^{i(x+t)} + \frac{i}{2}e^{ix}} + \text{c.c.}\right\}$$

$$= \left[\frac{1}{2}\sin(x+t) - \frac{1}{2}\sin(x)\right] + (\text{imaginary part})$$

$$\Downarrow$$

$$u(x,t) = \cosh(t) + \sin(x+t) - \sin(x)$$

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