Prob 1

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(n\pi x), \quad \underbrace{Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin(n\pi x)}$$

compare with the $Q(x,t)$ in PDE:

$q_1(t) = t$, $q_2(t) = e^{-t}$

$$\rightarrow \dot{a}_n = -(n\pi)^2 a_n + \pi^2 a_n + q_n(t)$$

From b.c.'s, only $n=1,2$ matter

$$\dot{a}_1 = q_1(t) = t \Rightarrow a_1(t) = a_1(0) + \frac{t^2}{2}$$

$$\dot{a}_2 = -3\pi^2 a_2 + q_2(t) = -3\pi^2 a_2 + e^{-t}$$

$$\Rightarrow a_2(t) = a_2(0)\, e^{-3\pi^2 t} + \underbrace{\int_0^t e^{-3\pi^2(t-\hat{t})} e^{-\hat{t}} d\hat{t}}_{= \frac{e^{-t} - e^{-3\pi^2 t}}{3\pi^2 - 1}}$$

From the b.c.'s at $t=0$: $a_1(0) = 1$, $a_2(0) = 0$

$$\Rightarrow \quad u(x,t) = a_1(t)\sin(\pi x) + a_2(t)\sin(2\pi x)$$

$$= \left(1 + \frac{t^2}{2}\right)\sin(\pi x) + \left(\frac{e^{-t} - e^{-3\pi^2 t}}{3\pi^2 - 1}\right)\sin(2\pi x)$$

⋇

Prob 2

The steady solution, $U(x)$, satisfies $\begin{cases} U'' = -U \\ U(0) = 1 \\ U'(\pi) = 1 \end{cases}$

$\Rightarrow U(x) = \cos(x) - \sin(x)$.

Let $u_*(x,t) \equiv u(x,t) - U(x)$

$$\Rightarrow \begin{cases} \dfrac{\partial u_*}{\partial t} = \dfrac{\partial^2 u_*}{\partial x^2} + u_* \\ u_*(0,t) = 0 \\ u_{*_x}(\pi,t) = 0 \\ u_*(x,0) = u(x,0) - U(x) = \sin\left(\frac{x}{2}\right) \end{cases}$$

By the standard method for a homogeneous system,

$$u_*(x,t) = \sum_{\substack{n=1 \\ \langle \text{odd } n \rangle}}^{\infty} a_n e^{\left(1 - \frac{n^2}{4}\right)t} \sin\left(\frac{nx}{2}\right)$$, Sum over <u>odd</u> n only

From the 3rd b.c. for u_*, only $n=1$ matters, and $a_1 = 1$.

$\rightarrow u_*(x,t) = e^{\frac{3}{4}t} \sin\left(\frac{x}{2}\right)$

$\rightarrow$ Full solution is

$$u(x,t) = u_*(x,t) + U(x) = \cos(x) - \sin(x) + e^{\frac{3}{4}t} \sin\left(\frac{x}{2}\right)$$

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Prob 3

The steady solution $U(x)$ satisfies $\begin{cases} U'' = -U - 2019 \\ U'(0) = 1 \\ U'(\frac{\pi}{2}) = 0 \end{cases}$

$\Rightarrow U(x) = \sin(x) - 2019$

Can be easily solved by the change of var.: $V \equiv U + 2019$

Let $u_*(x,t) \equiv u(x,t) - U(x)$

$$\Rightarrow \begin{cases} \dfrac{\partial u_*}{\partial t} = \dfrac{\partial^2 u_*}{\partial x^2} + u_* \\ u_{*_x}(0,t) = 0 \\ u_{*_x}(\frac{\pi}{2}, t) = 0 \\ u_*(x,0) = \sin(x) - U(x) = 2019 \end{cases}$$

$$\Rightarrow \quad u_*(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos(2nx) \quad \text{—— } (\cancel{*})$$

From the 3rd b.c. for u_*, only $\underline{n=0}$ matters.

The equation for $a_0(t)$ is

$$\frac{da_0}{dt} = a_0 \quad \Rightarrow \quad a_0(t) = a_0(0)e^t$$

$$\Rightarrow \quad u_*(x,t) = a_0(t) = a_0(0)e^t$$

From $\underline{\text{b.c. } 3}$ for u_*, $a_0(0) = 2019$

$$\Rightarrow \quad u_*(x,t) = 2019\, e^t$$

$\Rightarrow$ Full solution is

$$u(x,t) = u_*(x,t) + U(x) = \sin(x) + 2019(e^t - 1)$$

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Prob 4

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}, \quad \underbrace{Q(x,t) = \sum_{n=-\infty}^{\infty} q_n(t) e^{inx}}$$

$$\rightarrow \quad \ddot{C}_n = (1-n^2) C_n + q_n$$

From $Q(x,t)$ and the b.c.'s, only $n=0, \pm 1, \pm 3$ matter

by visual inspection: $q_0(t) = 5 \quad q_1(t) = \frac{t}{2i}$
$q_3(t) = 0$

$$\ddot{C}_0 = C_0 + q_0 = C_0 + 5 \Rightarrow C_0(t) = A_0 \cosh(t) + B_0 \sinh(t) - 5$$

$$\rightarrow \dot{C}_0(t) = A_0 \sinh(t) + B_0 \cosh(t)$$

$$\ddot{C}_1 = q_1(t) = \frac{t}{2i} \Rightarrow C_1(t) = C_1(0) + \dot{C}_1(0)t + \frac{t^3}{12i}$$

$$\ddot{C}_3 = -8C_3 \Rightarrow C_3(t) = A_3 \cos(\sqrt{8}t) + B_3 \sin(\sqrt{8}t)$$

$$\rightarrow \dot{C}_3(t) = -\sqrt{8} A_3 \sin(\sqrt{8}t) + \sqrt{8} B_3 \cos(\sqrt{8}t)$$

$$u(x,t) = C_0(t) + \left\{ \left(C_1(t) e^{ix} + C_3(t) e^{i3x} \right) + c.c. \right\}$$

From b.c. (i) $\rightarrow$ $C_0(0) = 1$, $C_1(0) = 0$, $C_3(0) = 0$

b.c. (ii) $\rightarrow$ $\dot{C}_0(0) = 0$, $\dot{C}_1(0) = 0$, $\dot{C}_3(0) = \frac{1}{2}$

$\Downarrow$ $A_0 = 6, \; B_0 = 0$

$\Downarrow$ $A_3 = 0, \; B_3 = \frac{1}{4\sqrt{2}}$

$$\Rightarrow u(x,t) = \left(6\cosh(t) - 5\right) + \left\{ \left(\frac{-i}{12} t^3 e^{ix} + \frac{1}{2\sqrt{8}} \sin(\sqrt{8}t) e^{i3x} \right) + c.c. \right\}$$

$$= 6\cosh(t) - 5 + \frac{t^3}{6}\sin(x) + \frac{1}{\sqrt{8}} \sin(\sqrt{8}t)\cos(3x)$$

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