Prob 2

sep. of var. $u \sim G(x)H(t)$

$$\Rightarrow (1+t)^2 \frac{\dot H}{H} - 4\pi^2 = \boxed{\frac{G''}{G} = C \qquad \begin{matrix} G(0)=0 \\ G(1)=0 \end{matrix}}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\frac{\dot H}{H} = \frac{(4-n^2)\pi^2}{(1+t)^2} \Longleftarrow \begin{matrix} C_n = -(n\pi)^2 \\ G_n(x) = \sin(n\pi x) \end{matrix}, \quad n = 1, 2, 3, \dots$$

Observing the 3rd b.c., only $n = 2, 3$ matter.

$\underline{n=2}$: $\dot H_2 = 0 \Rightarrow H_2(t) = \text{constant} \leftarrow$ set to 1

$\underline{n=3}$: $\dfrac{\dot H_3}{H_3} = \dfrac{-5\pi^2}{(1+t)^2} \Rightarrow H_3(t) = e^{\frac{5\pi^2}{1+t}}$

$\uparrow$ set leading constant to 1

Full solution:

$$u(x,t) = a_2 G_2(x) H_2(t) + a_3 G_3(x) H_3(t)$$
$$= a_2 \sin(2\pi x) + a_3 \sin(3\pi x)\, e^{\frac{5\pi^2}{1+t}}$$

compare with b.c. (iii)

$$\Rightarrow \quad a_2 = 1, \quad a_3 = 2e^{-5\pi^2}$$

$\Rightarrow$ Full solution is

$$u(x,t) = \sin(2\pi x) + 2\sin(3\pi x)\, e^{-5\pi^2\left(1 - \frac{1}{1+t}\right)}$$

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At $t \to \infty$, $\frac{1}{1+t} \to 0$, and the steady solution is

$$u_s(x) = u(x,\infty) = \sin(2\pi x) + 2\sin(3\pi x)\, e^{-5\pi^2}$$

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Prob 3

Sep. of var. $u \sim G(x)H(t)$

$$\Rightarrow \quad \frac{1}{1+t}\left[\frac{\dot{H}}{H} - \sin(t)\right] = \boxed{\frac{G''}{G} = c \qquad \begin{matrix} G'(0)=0 \\ G'(100)=0 \end{matrix}}$$

$$c = 0, \left\{-\left(\frac{n\pi}{100}\right)^2, \ n = 1, 2, 3, \dots\right\}$$

$$G_0(x) = 1, \quad G_n(x) = \cos\left(\frac{n\pi x}{100}\right)$$

Observing b.c. (iii), only $n=0$ and 200 matter.

$\underline{n=0}$: $\frac{1}{1+t}\left[\frac{\dot{H}_0}{H_0} - \sin(t)\right] = 0 \Rightarrow \frac{\dot{H}_0}{H_0} = \sin(t) \Rightarrow H_0(t) = e^{-\cos(t)}$

constant set to 1

$\underline{n=200}$: $\frac{1}{1+t}\left[\frac{\dot{H}_{200}}{H_{200}} - \sin(t)\right] = c_{200} = -4\pi^2$

$$\Rightarrow \frac{\dot{H}_{200}}{H_{200}} = -4\pi^2(1+t) + \sin(t) \Rightarrow H_{200}(t) = e^{-4\pi^2\left(t+\frac{t^2}{2}\right) - \cos(t)}$$

constant set to 1

Full solution:

$$u(x,t) = a_0 G_0(x) H_0(t) + a_{200} G_{200}(x) H_{200}(t)$$
$$= a_0 e^{-\cos(t)} + a_{200} \cos(2\pi x) e^{-4\pi^2\left(t+\frac{t^2}{2}\right) - \cos(t)}$$

compare with b.c. (iii) $\Rightarrow a_0 = e, \ a_{200} = e.$

Full solution is

$$u(x,t) = e^{1-\cos(t)} + \cos(2\pi x) e^{-4\pi^2\left(t+\frac{t^2}{2}\right) + 1 - \cos(t)}$$

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