Prob 2

(a) The system has multiple solutions.

(b) Sep. of var. $\Rightarrow$ $\quad -\dfrac{G''}{G} = \boxed{\dfrac{\ddot{H}}{H} = c \qquad \begin{matrix} \dot{H}(0) = 0 \\ \dot{H}(1) = 0 \end{matrix}}$

$u \sim G(x)H(y)$

$\Downarrow$

$c = 0, \{-(n\pi)^2 \ \ n = 1, 2, 3, \dots\}$

$\cancel{G_0(x)} \Rightarrow H_0(y) = 1, \quad H_n(y) = \cos(n\pi y)$

$n = 1, 2, 3, \dots$

$n = 0, \quad G_0'' = 0$

$\Rightarrow G_0(x) = A_0 x + B_0$

$n > 0, \quad G_n'' = (n\pi)^2 G_n$

$\Rightarrow G_n(x) = A_n \cosh(n\pi x) + B_n \sinh(n\pi x)$

Observing b.c. (i) and (ii), only $n = 0, 3$ matter.

Full solution is $\quad u(x, y) = a_0 G_0(x) H_0(y) + a_3 G_3(x) H_3(y)$

$\rightarrow \quad u(x, y) = a_0 x + b_0 + a_3 \cosh(3\pi x)\cos(3\pi y) + b_3 \sinh(3\pi x)\cos(3\pi y)$

$\Rightarrow \quad u_x(x, y) = a_0 + 3\pi a_3 \sinh(3\pi x)\cos(3\pi y) + 3\pi b_3 \cosh(3\pi x)\cos(3\pi y)$

b.c. (i) : $\quad 5 + \cos(3\pi y) = a_0 + 3\pi b_3 \cos(3\pi y) \Rightarrow a_0 = 5, \ b_3 = \dfrac{1}{3\pi}$

b.c. (ii) : $\quad 5 = 5 + [3\pi a_3 \sinh(3\pi) + \cosh(3\pi)] \cos(3\pi y)$

$\Rightarrow 3\pi a_3 \sinh(3\pi) + \cosh(3\pi) = 0 \Rightarrow a_3 = -\dfrac{\cosh(3\pi)}{3\pi \sinh(3\pi)}$

b_0 remains undetermined.

$\Rightarrow$ Full solution is

$$u(x, y) = 5x + b_0 + \frac{1}{3\pi}\left[\sinh(3\pi x) - \frac{\cosh(3\pi)}{\sinh(3\pi)} \cosh(3\pi x)\right] \cos(3\pi y)$$

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since b_0 can be of any value, there are infinite many solutions.

Prob 3 (Corrected)

Sep. of var. $\Rightarrow$ $G''H + y^2 G\ddot{H} - yG\dot{H} + GH = 0$ $\boxed{\begin{array}{l} G(0)=0 \\ G(\pi)=0 \end{array}}$

$u \sim G(x)H(y)$ $\Rightarrow$ $\dfrac{-y^2\ddot{H} + y\dot{H} - H}{H} = \boxed{\dfrac{G''}{G} = c \quad \begin{array}{l} \cancel{G'(0)=0} \\ \cancel{G'(\pi)=0} \end{array}}$

$\Downarrow$

$c = \cancel{0}, \{-n^2, n=1,2,3,\dots\}$

$\cancel{G_0(x)=0}$

$G_n(x) = \cancel{\cos(nx)}\ \cancel{\sin(n\pi)}\ \sin(nx)$

Observing b.c. (iii), (iv), only $n=3$ will survive in full solution

$\hookrightarrow$ suffices to solve $H_3(y)$

$\rightarrow \quad -y^2\ddot{H}_3 + y\dot{H}_3 - H_3 = c_3 H_3 = -9H_3$

$\rightarrow \quad y^2\ddot{H}_3 - y\dot{H}_3 - 8H_3 = 0$ (This is a "Cauchy" ODE)

Let $H_3(y) \sim y^p$

$\Rightarrow \quad p^2 - 2p - 8 = 0 \Rightarrow p = 4, -2$

$\Rightarrow \quad H_3(y) = A y^4 + \dfrac{B}{y^2}$

Full solution is $u(x,y) = a_3 G_3(x) H_3(y)$

$= \left(A_3 y^4 + \dfrac{B_3}{y^2}\right) \cancel{\cos(3x)}\ \sin(3x)$

$$\left.\begin{array}{ll} \text{b.c. (iii)}: & A_3 + B_3 = 5 \\ \text{b.c. (iv)}: & 16A_3 + \dfrac{B_3}{4} = 17 \end{array}\right\} \Rightarrow A_3 = 1, \ B_3 = 4$$

Full solution: $u(x,y) = \left(y^4 + \dfrac{4}{y^2}\right) \cancel{\cos(3x)}\ \sin(3x)$

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Prob 4

Sep. of var. $\rightarrow$ $\dfrac{\ddot{H}}{H} - 1 = \boxed{\dfrac{G''}{G} = c \qquad G'(0)=0, \quad G'(\pi)=0}$

$u \sim G(x)H(t)$

$\Downarrow$

$c = 0, \{-n^2, n = 1, 2, 3, \dots\}$

$G_0(x) = 0, \quad G_n(x) = \cos(nx),$
$n = 1, 2, 3, \dots$

$$\ddot{H}_n = (1+c) H_n$$
$$= (1-n^2) H_n$$

Observing b.c. (iii) and (iv), only $n = 0, 1,$ and 2 matter.

$n = 0 \Rightarrow \ddot{H}_0 = H_0, \quad H_0(t) = A_0 \cosh(t) + B_0 \sinh(t)$

$n = 1 \Rightarrow \ddot{H}_1 = 0, \quad H_1(t) = A_1 t + B_1$

$n = 2 \Rightarrow \ddot{H}_2 = -3H_2, \quad H_2(t) = A_2 \cos(\sqrt{3}\, t) + B_2 \sin(\sqrt{3}\, t)$

Full solution:

$$u(x,t) = a_0 G_0(x) H_0(t) + a_1 G_1(x) H_1(t) + a_2 G_2(x) H_2(t)$$
$$= A_0 \cosh(t) + B_0 \sinh(t) + (A_1 t + B_1)\cos(x) + \left[A_2 \cos(\sqrt{3}t) + B_2 \sin(\sqrt{3}t)\right]\cos(2x).$$

$$\rightarrow u_t(x,t) = A_0 \sinh(t) + B_0 \cosh(t) + A_1 \cos(x) + \left[(-\sqrt{3}A_2)\sin(\sqrt{3}t) + \sqrt{3}B_2 \cos(\sqrt{3}t)\right]\cos(2x).$$

b.c. (iii): $1 = A_0 + B_1 \cos(x) + A_2 \cos(2x) \Rightarrow A_0 = 1, \ B_1 = 0, \ A_2 = 0$

b.c. (iv): $\cos(x) + \cos(2x) = B_0 + A_1 \cos(x) + \sqrt{3} B_2 \cos(2x)$

$$\Rightarrow B_0 = 0, \ A_1 = 1, \ B_2 = \frac{1}{\sqrt{3}}$$

Full solution:

$$u(x,t) = \cosh(t) + t\cos(x) + \frac{1}{\sqrt{3}} \sin(\sqrt{3}\, t)\cos(2x)$$

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