Prob 2

F.S. expansion : $u(x,t) = \sum_n C_n(t) e^{inx}$

$$\Rightarrow \quad \frac{dC_n}{dt} = [-4n^2 t - in^3 t + (1+t)n^4 - 16]\, C_n$$

observing the b.c., only $n = 0, \pm 2$ matter

$n=0 \Rightarrow \quad \dot{C}_0 = -16 C_0 \quad C_0(t) = C_0(0) e^{-16t} = e^{-16t}$

$n=2 \Rightarrow \quad \dot{C}_2 = -i8t C_2 \quad C_2(t) = C_2(0) e^{-i4t^2} = \frac{1}{2i} e^{-i4t^2}$

From b.c. (i)

$$\sum_n C_n(0) e^{inx} = 1 + \sin(2x) = 1 + \left[\frac{1}{2i} e^{i2x} + c.c.\right]$$

$$\Rightarrow C_0(0) = 1$$
$$C_2(0) = \frac{1}{2i}$$

Full solution :

$$u(x,t) = C_0(t) + [C_2(t) e^{i2x} + c.c.]$$
$$= e^{-16t} + \left[\frac{1}{2i} e^{i(2x-4t^2)} + c.c.\right]$$
$$= e^{-16t} + \sin(2x - 4t^2) \quad \#$$

$$\frac{1}{2i} e^{i(2x-4t^2)} + c.c.$$
$$= \frac{-i}{2}\left[\cos(2x-4t^2) + i\sin(2x-4t^2)\right] + c.c.$$
$$= \sin(2x-4t^2)$$

Prob 3

F.S. expansion : $u(x,t) = \sum_n C_n(t) e^{inx}$

$$\frac{d^2 C_n}{dt^2} = (-9n^2 + n^4 + 8)\, C_n$$

From the b.c., only $n = 1, 3$ (and $-1, -3$) matter

$n=1: \quad \ddot{C}_1 = 0 \quad C_1(t) = A_0 t + B_0$

$C_1(0) = 0 \Rightarrow B_0 = 0 \quad \dot{C}_1(0) = \frac{1}{2i} \Rightarrow A_0 = \frac{1}{2i}$

$n=3: \quad \ddot{C}_3 = 8C_3 \quad C_3(t) = A_3 \cosh(\sqrt{8}t) + B_3 \sinh(\sqrt{8}t)$

$\quad \dot{C}_3(t) = \sqrt{8} A_3 \sinh(\sqrt{8}t) + \sqrt{8} B_3 \cosh(\sqrt{8}t)$

From b.c. (i) :

$$\sum_n C_n(0) e^{inx} = \cos(3x) = \frac{1}{2} e^{i3x} + c.c.$$

$$\Rightarrow C_3(0) = \frac{1}{2}, \; C_1(0) = 0$$

From b.c. (ii) :

$$\sum_n \dot{C}_n(0) e^{inx} = \sin(x) = \frac{1}{2i} e^{ix} + c.c.$$

$$\Rightarrow \dot{C}_1(0) = \frac{1}{2i}, \; \dot{C}_3(0) = 0$$

$C_3(0) = \frac{1}{2} \Rightarrow A_3 = \frac{1}{2}, \quad \dot{C}_3(0) = 0 \Rightarrow B_3 = 0$

Full solution : $u(x,t) = [C_1(t) e^{ix} + C_3(t) e^{i3x}] + c.c.$

$$= \left[\frac{t}{2i} e^{ix} + \frac{1}{2} e^{i3x} \cosh(\sqrt{8}t)\right] + c.c.$$
$$= t\sin(x) + \cos(3x)\cosh(\sqrt{8}\,t)$$

$\#$