Prob 1

Sep. of var. on the homogeneous sub-system:

$u \sim G(x)H(t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
$$u_x(0,t)=0$$
$$u_x(1,t)=0$$

$$G_0(x) = 1,$$
$$n \neq 0 \Rightarrow G_n(x) = \cos(n\pi x)$$

$$Q(x,t) = t e^{-4\pi^2 t}\cos(2\pi x) + \sin(t)$$

$\Rightarrow$ Expand the full solution as

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(n\pi x)$$

Expand the forcing, $Q(x,t)$ in the same manner:

$$Q(x,t) = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\cos(n\pi x)$$

plugging into PDE: $$\frac{da_n}{dt} = -(n\pi)^2 a_n + q_n$$

Visual inspection: $q_0(t) = \sin(t)$, $q_2(t) = te^{-4\pi^2 t}$, all other $q_n = 0$

This, combined with b.c. (iii), indicates that only $n=0,2$ matter.

$n=0$: $\quad \frac{da_0}{dt} = q_0 \Rightarrow \frac{da_0}{dt} = \sin(t) \quad a_0(t) = a_0(0) + 1 - \cos(t)$

$n=2$: $\quad \frac{da_2}{dt} = -4\pi^2 a_2 + q_2 = -4\pi^2 a_2 + te^{-4\pi^2 t}$

$$\Rightarrow a_2(t) = a_2(0)e^{-4\pi^2 t} + \int_0^t \hat{t}e^{-4\pi^2\hat{t}}e^{-4\pi^2(t-\hat{t})}d\hat{t}$$
$$= a_2(0)e^{-4\pi^2 t} + e^{-4\pi^2 t}\int_0^t \hat{t}\,d\hat{t}$$
$$= \left(a_2(0) + \frac{t^2}{2}\right)e^{-4\pi^2 t}$$

Visual inspection: $a_0(0) = 0$, $a_2(0) = 1$

$\Rightarrow$ Full solution $\quad u(x,t) = a_0(t) + a_2(t)\cos(2\pi x)$

$$= 1 - \cos(t) + \left(1 + \frac{t^2}{2}\right)e^{-4\pi^2 t}\cos(2\pi x)$$

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Prob 2

stead solution $U(x)$ is governed by

$$\boxed{\begin{array}{l} \dfrac{d^2U}{dx^2} - \cos(x) = 0 \\ U(0) = 0, \quad U(\pi) = 2 \end{array}}$$

solve:

$$U(x) = -\cos(x) + Ax + B$$

$U(0)=0 \Rightarrow B=1$, $U(\pi)=2 \Rightarrow A=0$

$$\Rightarrow U(x) = 1 - \cos(x)$$

Let $\hat{u}(x,t) = u(x,t) - U(x)$

$$\Rightarrow \boxed{\frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2} \qquad \begin{array}{l} \hat{u}(0,t) = 0 \\ \hat{u}(\pi,t) = 0 \\ \hat{u}(x,0) = \sin(x) \end{array}}$$

$\Downarrow$

solve: $\hat{u}(x,t) = \sin(x)\, e^{-t}$

$\Rightarrow$ Full solution is

$$u(x,t) = \hat{u}(x,t) + U(x) = 1 - \cos(x) + \sin(x)\, e^{-t}$$

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Prob 3

periodic b.c. in x $\Rightarrow$ Expand $u(x,t)$ in F.S.

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$ plugging into PDE

$$\Rightarrow \quad \dot{C}_n = (n^4 - n^2) c_n + q_n$$

From the inspection of $Q(x,t)$ and b.c. (ii), only $n = 1, 2$ matter (of course, we also keep $n = -1, -2$, but those components never need to be processed.)

$$Q(x,t) = \sum_{n=-\infty}^{\infty} q_n(t) e^{inx}$$

$$Q(x,t) = \sin(x)\sin(t) + \cos(2x)$$

$$= \frac{1}{2i}\sin(t) e^{ix} + \frac{1}{2} e^{i2x} + (\text{complex conjugate})$$

$$\Rightarrow q_1(t) = \frac{1}{2i}\sin(t), \quad q_2 = \frac{1}{2},$$

all other $q_n = 0$

$n=1$ $\quad \dot{C}_1 = q_1 = \frac{1}{2i}\sin(t)$

$$\Rightarrow C_1(t) = C_1(0) + \frac{1}{2i}[1 - \cos(t)]$$

$n=2$ $\quad \dot{C}_2 = 12 C_2 + q_2 = 12 C_2 + \frac{1}{2}$

$$\Rightarrow \quad C_2(t) = C_2(0) e^{12t} + \int_0^t \frac{1}{2} e^{12(t-\hat{t})} d\hat{t}$$

$$= C_2(0) e^{12t} + \frac{1}{2} e^{12t} \left(\frac{1 - e^{-12t}}{12}\right)$$

$$u(x,0) = \sum_n C_n(0) e^{inx}$$
$$= \cos(x) = \frac{1}{2}e^{ix} + \text{c.c.}$$
$$\Downarrow$$
$C_1(0) = \frac{1}{2}$, all other $C_n(0) = 0$

visual inspection: $C_1(0) = \frac{1}{2}, \quad C_2(0) = 0$

$\Rightarrow$ Full solution:

$$u(x,t) = C_1(t)e^{ix} + C_2(t)e^{i2x} + \text{c.c.}$$

$$= \left[\frac{1}{2} - \frac{i}{2}(1 - \cos(t))\right] e^{ix} + \frac{1}{2}\left(\frac{e^{12t} - 1}{12}\right) e^{i2x} + \text{c.c.}$$

$$= \cos(x) + [1 - \cos(t)]\sin(x) + \left(\frac{e^{12t} - 1}{12}\right)\cos(2x)$$

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$$\text{Re}\left\{\left[\frac{1}{2} - \frac{i}{2}(1 - \cos(t))\right] e^{ix}\right\}$$

$$= \text{Re}\left\{\left[\frac{1}{2} - \frac{i}{2}(1 - \cos(t))\right]\left[\cos(x) + i\sin(x)\right]\right\}$$

$$= \frac{1}{2}\cos(x) + \frac{1}{2}[1 - \cos(t)]\sin(x)$$

Prob 4

periodic b.c. in x $\Rightarrow$ Expand $u(x,t)$ in F.S., $u(x,t) = \sum_n C_n(t) e^{inx}$

Also, expand $Q(x,t)$ in F.S.

plugging into PDE

$$\ddot{C}_n = -n^2 C_n + 9 C_n + q_n$$

Inspecting $Q(x,t)$ and b.c. (i), (ii), only $n = 0, 3$ matter.

$$Q(x,t) = 9 + t \sin(3x) = 9 + \left[\frac{t}{2i} e^{i3x} + c.c.\right]$$

$$Q(x,t) = \sum_n q_n(t) e^{inx}$$

$\Rightarrow$ $q_0(t) = 9$,

$q_3(t) = \frac{t}{2i}$

all other $q_n(t) = 0$

From b.c. (i)

$$\sum_n C_n(0) e^{inx} = 0$$

$\Rightarrow$ All $C_n(0) = 0$

From b.c. (ii)

$$\sum_n \dot{C}_n(0) e^{inx} = 1$$

$\Rightarrow$ $\dot{C}_0(0) = 1$,

all other $\dot{C}_n(0) = 0$

$n = 0 \Rightarrow \ddot{C}_0 = 9 C_0 + 9$

Let $\hat{C}_0 \equiv C_0 + 1 \Rightarrow \ddot{\hat{C}}_0 = 9 \hat{C}_0$

$\Rightarrow \hat{C}_0(t) = A \cosh(3t) + B \sinh(3t)$

$\Rightarrow C_0(t) = A \cosh(3t) + B \sinh(3t) - 1$

$C_0(0) = 0 \Rightarrow A = 1$, $\dot{C}_0(0) = 1 \Rightarrow B = \frac{1}{3}$

$$C_0(t) = \cosh(3t) + \frac{1}{3}\sinh(3t) - 1$$

$n = 3 \Rightarrow \ddot{C}_3 = q_3(t) = \frac{t}{2i}$

$$\Rightarrow C_3(t) = C_3(0) + \dot{C}_3(0)\, t + \frac{t^3}{12i} = \frac{t^3}{12i}$$

(since $C_3(0) = 0$, $\dot{C}_3(0) = 0$)

Full solution:

$$u(x,t) = C_0(t) + \left[C_3(t) e^{i3x} + c.c.\right]$$

$$= \cosh(3t) + \frac{1}{3}\sinh(3t) - 1 + \frac{t^3}{6}\sin(3x)$$

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$$Re\left\{C_3(t) e^{i3x}\right\}$$

$$= Re\left\{\frac{-i}{12} t^3 \left(\cos(3x) + i \sin(3x)\right)\right\}$$

$$= \frac{1}{12} t^3 \sin(3x)$$