MAE/MSE 502, Spring 2020 Homework # 3

Hard copy of report is due 6:00 PM on the due date. The report should include a statement on collaboration, and computer code(s) used for the assignment. See the first page of the problem sheet for Homework #1 for the rules on collaboration.

Prob 1 (2 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 8$ and $t \ge 0$, consider the 1-D Wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(8, t) = 0$, (iii) $u(x, 0) = P(x)$, (iv) $u_t(x, 0) = 0$.

(a) Solve the system with P(x) given as

$$\begin{aligned} P(x) &= x/2 \quad &&, \text{if } 0 \le x \le 2 \\ &= (8 - x)/6 &&, \text{if } 2 < x \le 8 , \end{aligned}$$

and plot the solution as a function of x at t = 0, 2.4, 4.0, 5.6, 8.0, and 14.4. Please collect all 6 curves in one plot.

(b) Solve the system with $P(x)$ given as (this emulates a "wave packet")

$$P(x) = \sum_{n=30}^{50} \exp\left[-\left(\frac{n-40}{4}\right)^2\right] \sin\left(\frac{n\pi x}{8}\right) ,$$

and plot the solution as a function of x at t = 0, 2, 4, and 8. For this part, it is recommended that the four curves be plotted separately, for example arranged in 4 panels using "subplot" in Matlab.

Prob 2 (1.5 points)
Consider the eigenvalue problem for $G(x)$ defined on the interval, $0 \le x \le 1$,

$$\frac{d^2 G}{dx^2} = cG, \; G'(0) = 2 , \; G'(1) = 2 .$$

(G' is the derivative of G. Note that both b.c.'s are imposed on G'.)
(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem. Are the eigenvalues discrete? For example, if the boundary conditions are replaced by the familiar $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2 \pi^2$ (n is an integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values within an interval on the c-axis are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

(b) Plot the eigenfunctions, $G_c(x)$, associated to the eigenvalues c = –50, –5, 5, and 50. (You will find in Part (a) that all those values are indeed valid eigenvalues.) Please collect all 4 curves in a single plot and label them clearly. Since $G(x)$ is defined only on the interval of $0 \le x \le 1$, the plot should cover only that interval.

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x) G_q(x)\, dx = 0, if p \neq q,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions associated to two distinctive eigenvalues p and q? Your answer should be more than just "yes" or "no". In order to claim that the orthogonality relation does not hold, it suffices to produce a counterexample by finding a pair of eigenfunctions (associated to two distinctive eigenvalues) for which the integral of $G_p(x)G_q(x)$ is non-zero. On the other hand, to claim that the orthogonality relation holds, one must show that it holds for all possible pairs of p and q.

Prob 3 (2.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, consider the PDE (in which U and K are constants)

$$\frac{\partial u}{\partial t} = U \frac{\partial u}{\partial x} + K \frac{\partial^2 u}{\partial x^2}$$

with periodic boundary conditions in the x-direction. The boundary conditions in the t-direction are given as

$$u(x, 0) = \frac{[1 - \cos(x)]^{10}}{1024} .$$

Solve the PDE by Fourier series expansion. Plot the solution $u(x, t)$ at $t = 0.1$ for the three cases with

(i) $U = 10, K = 0$

(ii) $U = 0, K = 2$

(iii) $U = 10, K = 2$.

Also, plot the solution at $t = 0$ (which is the same for all three cases). Please collect all four curves in one plot.

For this problem, the solution can be expressed as an infinite series. To compute the values of u for the plot, the series can be truncated to a finite number of terms (see relevant remarks below HW1-Prob1), and Matlab can be used to evaluate the expansion coefficients by numerical integration.

Prob. 4 (1.5 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^5 u}{\partial x^5} + \frac{\partial^3 u}{\partial x^3} + \left(\frac{1}{1+t}\right)\frac{\partial^2 u}{\partial x^2} + \left(\frac{4}{1+t}\right)u\,,$$

with periodic boundary conditions in the x-direction. The boundary condition in the t-direction is given as

$u(x, 0) = 1 + \sin(x) + \cos(2x)$.

We expect a closed-form solution which consists of only a finite number of terms and no unevaluated integrals. The solution should be expressed in real numbers and functions. Expect a deduction if an imaginary number "i" (=$\sqrt{-1}$) is left in the solution.

Prob 5 (1.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x\,\partial t} - 3\frac{\partial u}{\partial t} - 4\,u = 0\,,$$

with periodic boundary conditions in the x-direction, and the boundary conditions at $t = 0$ given as

(i) $u(x, 0) = \sin(2x)$

(ii) $u_t(x, 0) = 10$.

We expect a closed-form solution which consists of only a finite number of terms and no unevaluated integrals. The solution should be expressed in real numbers and functions. Expect a deduction if an imaginary number "i" (=$\sqrt{-1}$) is left in the solution.