MAE/MSE 502, Spring 2020 Homework #5

Please upload the report as a single pdf or doc/docx file to Canvas. The report should include a statement on collaboration. See the cover page of Homework #1 for the rules on collaboration. **For Prob 2-6 in this homework, we expect a closed-form solution without any unevaluated integrals. The solution, *u*(*x*, *t*), must be written explicitly as a function of *x* and *t*.**

Prob 1 (1 point)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, consider the system of *Poisson equation* (where the "source term" Q is a given function of x and y),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = Q \ ,$$

with the boundary conditions (u_x and u_y denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively),

(i) $u_x(0, y) = 0$ (ii) $u_x(1, y) = 0$ (iii) $u_y(x, 0) = 0$ (iv) $u_y(x, 1) = 2$.

Let $Q(x,y) = x^2 + K\,y$ where K is a constant. For what value(s) of K may a solution (or solutions) exist for the system?

Prob 2 (1 point)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \ge 0$, solve the PDE

$$(1+t)\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = tu$$

with the boundary condition

$$u(x,0) = e^{-x^2} \ .$$

Prob 3 (2.5 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \ge 0$, solve the PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = x$$

with the boundary condition,

$$u(x,0) = \begin{cases} 0\,, \ if\ x < 1 \\ x-1\ ,\ if\ x \ge 1 \end{cases}$$

What is the steady solution (as $t \to \infty$) for the system? Plot the solution, $u(x, t)$, as a function of x at $t = 0.3$ and 0.7. In addition, superimpose the initial state, $u(x, 0)$, and the steady solution, $u_s(x)$. The recommended range for plotting is $-1 \le x \le 3$. Please collect all four curves in a single plot.

Prob 4 (2.5 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} + t\, e^{u}\, \frac{\partial u}{\partial x} = e^{-u}$$

with the boundary condition,

$$u(x,0) = \begin{cases} 0\,,\ if\ x < 1 \\ ln(x)\ ,\ if\ x \geq 1 \end{cases}$$

($ln(x)$ is the natural logarithm of x.)
Plot the solution, $u(x, t)$, as a function of x at $t = 0.2$ and 0.8. In addition, superimpose the initial state, $u(x, 0)$. The recommended range for plotting is $-1 \leq x \leq 5$. Please collect all three curves in a single plot.

Prob 5 (2.5 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} + (1 + u)\, \frac{\partial u}{\partial x} = u$$

with the boundary condition,

$$u(x,0) = \begin{cases} 0\,,\ if\ x < 0 \\ x^2,\ if\ 0 \leq x \leq 1 \\ 1\,,\ if\ x > 1 \end{cases}$$

Plot the solution, $u(x, t)$, as a function of x at $t = 0.2$ and 0.5. In addition, superimpose the initial state, $u(x, 0)$. The recommended range for plotting is $-1 \leq x \leq 3$. Please collect all three curves in a single plot.

Prob 6 (2.5 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with the boundary conditions,

(i) $u(x,0) = 1$

(ii) $u_t(x,0) = x$.