Prob 1

The eigenvalues and corresponding eigenfunctions are:

$c = 0$, $G_0(x) = Ax$, where A is an arbitrary constant.

$c = -(6n)^2$, $n = 1, 2, 3, \ldots$, $G_n(x) = A \sin(6nx)$, where A is an arbitrary constant.

All $c > 0$ are not valid eigenvalues, as they lead to a trivial solution.

Prob 2

The full solution is

$$u(x,t) = \sin(\pi x)e^{(100-\pi^2)(1-e^{-t})}$$

A steady solution (at $t \to \infty$) exists as

$$u_s(x) = \sin(\pi x)e^{(100-\pi^2)}$$

Prob 3

(a) A unique solution exists as

$$u(x,y) = (3y+2)\cos(2\pi x)$$

(b) No solution, due to a contradiction. (Imposing the boundary conditions on the general solution would lead to "3 = 5".)

(c) The solution(s) are

$$u(x,y) = A\cos(2\pi y) + (3y+B)\cos(2\pi x)\ ,$$

where A and B are arbitrary constants. In other words, there are infinitely many solutions.

Note: For Prob 3(b) and (c), despite having Neumann boundary conditions, the existence and uniqueness of the solution(s) can only be determined by working out the detail of the full solution. Applying the "solvability condition" for Laplace's equation will not work, because the PDE is not a pure Laplace's equation. Recall that the original solvability condition for Laplace's equation is derived by integrating the PDE itself over the whole domain (*cf*. Lecture 11),

$$\iint \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\ dA = \iint 0\ dA\ ,$$

and further converting the left hand side of the equation to a line integral (by Gauss divergence theorem) along the circuit of the boundary, leading to

$$\oint (\nabla u)\cdot \hat{\boldsymbol{n}}\ dl = 0\ .$$

If the boundary conditions are of purely Neumann type, the line integral of the "gradient of u" can be fully determined by the boundary conditions imposed on the "flux". This leads to the solvability condition, for the pure Laplace's equation.

The argument breaks down when there is an extra term in the PDE like Prob 3 in the exam. By integrating the PDE over the whole domain as before, we find

$$\iint \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\, dA + 4\pi^2 \iint u\, dA = \iint 0 \; dA\,.$$

The first term in the L.H.S. can be further processed like we did for the pure Laplace's equation, but there is nothing further we can do to the second term. The final "solvability condition" for the system becomes

$$\oint (\nabla u) \cdot \hat{\boldsymbol{n}}\, dl + 4\pi^2 \iint u\, dA = 0 \;.$$

This is the correct "solvability condition" for the system in Prob 3, but it is mostly useless. The only way we can evaluate the second term in the L.H.S. is to know the full solution, $u(x,y)$, and honestly integrate it over the whole domain! Which is to say that the only way to settle the issue of solvability of the system is by the "brute-force" approach of seeking the full solution.

For Prob 3(a), the "solvability condition" does not work even for the pure Laplace's equation, since the boundary conditions are not of purely Neumann type.