Q2

Separation of var.: $u(x,t) \sim G(x)H(t)$

$$\Rightarrow \quad e^t \frac{\dot{H}}{H} - 4\pi^2 = \boxed{\frac{G''}{G} = c \quad \begin{matrix} G(0)=0 \\ G(1)=0 \end{matrix}} \Rightarrow \begin{matrix} c_n = -(n\pi)^2, \quad n=1,2,3,\ldots \\ G_n(x) = \sin(n\pi x) \end{matrix}$$

From b.c. (iii), only $n=2,3$ matter.

$$n=2: \quad e^t \frac{\dot{H}_2}{H_2} - 4\pi^2 = -4\pi^2 \Rightarrow \dot{H}_2 = 0 \Rightarrow H_2(t) = \underset{\uparrow}{H_2(0)} = \text{const.}$$

(OK to set to 1)

$$n=3: \quad e^t \frac{\dot{H}_3}{H_3} - 4\pi^2 = -9\pi^2$$

$$\Rightarrow \frac{\dot{H}_3}{H_3} = -5\pi^2 e^{-t} \Rightarrow H_3(t) = \underbrace{H_3(0)}_{\text{(OK to set to 1)}} e^{-5\pi^2(1-e^{-t})}$$

Full solution:

$$u(x,t) = a_2 G_2(x) H_2(t) + a_3 G_3(x) H_3(t)$$

$$= a_2 \sin(2\pi x) + a_3 \sin(3\pi x)\, e^{-5\pi^2(1-e^{-t})}$$

From b.c. (iii): $a_2 = 1$, $a_3 = 3$. (by visual inspection)

$$\Rightarrow \quad u(x,t) = \sin(2\pi x) + 3\sin(3\pi x)\, e^{-5\pi^2(1-e^{-t})}$$

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Taking the full solution and push $t \to \infty$, we obtain the steady solution as

$$u_s(x) = \sin(2\pi x) + 3\sin(3\pi x)\, e^{-5\pi^2}$$

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Q3

Separation of var.: $u(x,t) \sim G(x)H(t)$

$$\Rightarrow (t+t^2) + (1+t)\frac{\dot{H}}{H} = \boxed{\frac{G''}{G} = c \quad \begin{matrix} G'(0)=0 \\ G'(100)=0 \end{matrix}} \Rightarrow c = 0,\ -\left(\frac{n\pi}{100}\right)^2,\quad n=1,2,3,\dots$$

$$G_0(x) = 1$$

$$G_n(x) = \cos\left(\frac{n\pi x}{100}\right)$$

From b.c. (iii), only $n=0, 100$ matter.

$$n=0:\quad (t+t^2) + (1+t)\frac{\dot{H}_0}{H_0} = 0 \Rightarrow \frac{\dot{H}_0}{H_0} = -t \Rightarrow H_0(t) = \underbrace{H_0(0)}_{\text{(ok to set to 1)}} e^{-\frac{t^2}{2}}$$

$$n=100:\quad (t+t^2) + (1+t)\frac{\dot{H}_{100}}{H_{100}} = -\pi^2$$

$$\Rightarrow \frac{\dot{H}_{100}}{H_{100}} = \frac{-\pi^2}{1+t} - t \Rightarrow H_{100}(t) = H_{100}(0)\, e^{\int_0^t \left(\frac{-\pi^2}{1+t} - t\right)dt}$$

$$= \underbrace{H_{100}(0)}_{\text{(ok to set to 1)}} (1+t)^{-\pi^2} e^{-\frac{t^2}{2}}$$

Full solution:

$$u(x,t) = a_0 G_0(x) H_0(t) + a_{100} G_{100}(x) H_{100}(t)$$

$$= a_0 e^{-\frac{t^2}{2}} + a_{100} \cos(\pi x)(1+t)^{-\pi^2} e^{-\frac{t^2}{2}}$$

From b.c. (iii), by visual inspection, $a_0 = 1$, $a_{100} = 1$

$\Rightarrow$ Full solution: $$u(x,t) = e^{-\frac{t^2}{2}} + \cos(\pi x)(1+t)^{-\pi^2} e^{-\frac{t^2}{2}}$$

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