

MAE/MSE 502, Fall 2021 Homework #2

Please follow the rules on collaboration as given in the document for Homework #1. Your report should include a statement of collaboration.

Problem 1 (2.5 points)

For $u(x,y)$ defined on the square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with the boundary conditions,

$$(i) u(0, y) = 0 \quad (ii) u(1, y) = \sin(\pi\sqrt{y}) \quad (iii) u(x, 0) = 0 \quad (iv) u(x, 1) = 2(x - x^5) \quad .$$

Make a contour plot of the solution, $u(x, y)$. See Additional Note in the last page for an example of using Matlab to make a contour plot. The recommended contour interval for the plot is 0.1. For this problem, we expect the solution to be expressed as an infinite series. Please see the remark below HW1-Q1 on how to truncate the series and numerically compute the expansion coefficients.

Problem 2 (2.5 points)

For $u(x,y)$ defined on the square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + 9 \frac{\partial^2 u}{\partial y^2} = 0,$$

with the boundary conditions (u_x and u_y denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively)

$$(i) u_x(0, y) = 0 \quad (ii) u_x(1, y) = 0 \quad (iii) u_y(x, 0) = 2 \quad (iv) u_y(x, 1) = 2 + \cos(3\pi x)$$

(a) Test the solvability condition on the system. Based on it, which of the following is true?

(I) The system has no solution. (II) The system has a unique solution. (III) The system has multiple solutions. If your answer is (I), no need to proceed further. Otherwise, proceed to Part (b).

(b) Find the solution(s) of the system. If your answer for Part (a) is (III), please clearly write out what those multiple solutions are. For this part of the problem, we expect a closed-form analytic solution that consists of only a finite number of terms and without any unevaluated integrals. Expect a deduction if the requirement is not met.

Problem 3 (2.5 points)

For $u(x,y)$ defined on the rectangular domain of $0 \leq x \leq 1$ and $0 \leq y \leq \pi$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 5 \frac{\partial u}{\partial x} + 4u = 0,$$

with the boundary conditions

$$(i) u(0, y) = 1 + \cos(2y) \quad (ii) u(1, y) = e^4 + e^5 \cos(2y) \quad (iii) u_y(x, 0) = 0 \quad (iv) u_y(x, \pi) = 0$$

For this problem, we expect a closed-form analytic solution that consists of only a finite number of terms and without any unevaluated integrals. Expect a deduction if the requirement is not met.

Problem 4 (2.5 points)

For $u(x,y)$ defined on the rectangular domain of $0 \leq x \leq \pi$ and $0 \leq y \leq 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0,$$

with the boundary conditions

$$(i) u_x(0, y) = 0 \quad (ii) u_x(\pi, y) = 0 \quad (iii) u(x, 0) = \cos(x) \quad (iv) u(x, 1) = 1 + 2 \cos(x) + \cos(2x) \quad .$$

(Note that the first two boundary conditions are imposed on the derivative of u , while the 3rd and 4th are imposed on u .) For this problem, we expect a closed-form analytic solution that consists of only a finite number of terms and without any unevaluated integrals. Expect a deduction if the requirement is not met.

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x) + x \cos(3\pi y)$ over the domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, using the contour interval of 0.2 over the range of $-2 \leq u \leq 2$. Selected contours with $u = -1.6 + 0.4p$, $p = 1, 2, 3, \dots$, are labeled. The contour plot is shown in the next page.

It is essential to provide the coordinates of the grid ($x2d$ and $y2d$ in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y . Be aware that the setting used in this example might not be what is recommended for the actual contour plot in Problem 1. For that problem, please adjust the contour interval as needed.

```
x = [0:0.01:1]; y = [0:0.01:1];
for m = 1:length(x)
    for n = 1:length(y)
        u(m,n) = sin(2*pi*x(m)) + x(m) * cos(3*pi*y(n));
        x2d(m,n) = x(m); y2d(m,n) = y(n);
    end
end
[C h] = contour(x2d,y2d,u, [-2:0.2:2], 'LineWidth', 2)
clabel(C,h, [-1.6:0.4:1.6]); xlabel('x'); ylabel('y')
```

