Q2b

Sep. of var. : $u(x,y) \sim G(x)H(y)$

$$\Rightarrow \quad -9\frac{\ddot{H}}{H} = \boxed{\frac{G''}{G} = c \qquad \begin{matrix} G'(0)=0 \\ G'(1)=0 \end{matrix}} \Rightarrow \begin{matrix} c = 0, \; -(n\pi)^2 \quad n=1,2,3,\dots \\ G_0(x)=1, \quad G_n(x) = \cos(n\pi x) \end{matrix}$$

From b.c. (iii) and (iv), only $n=0, 3$ matter.

$n=0$: $\ddot{H}_0 = 0 \Rightarrow H_0(y) = A_0 y + B_0$

$n=3$: $-9\frac{\ddot{H}_3}{H_3} = -9\pi^2 \Rightarrow \ddot{H}_3 = \pi^2 H_3 \Rightarrow H_3(y) = A_3\cosh(\pi y) + B_3 \sinh(\pi y)$

Full solution: $u(x,y) = (A_0 y + B_0) + [A_3\cosh(\pi y) + B_3\sinh(\pi y)]\cos(3\pi x)$

$\hookrightarrow u_y(x,y) = A_0 + [\pi A_3 \sinh(\pi y) + \pi B_3\cosh(\pi y)]\cos(3\pi x)$

From b.c. (iii): $A_0 + \pi B_3 \cos(3\pi x) = 2 \Rightarrow A_0 = 2, \; B_3 = 0$

From b.c. (iv): $2 + \pi A_3 \sinh(\pi)\cos(3\pi x) = 2 + \cos(3\pi x)$

$$\Rightarrow A_3 = \frac{1}{\pi\sinh(\pi)}$$

B_0 remains undetermined.

Full solution:

$$u(x,y) = B_0 + 2y + \frac{\cosh(\pi y)\cos(3\pi x)}{\pi\sinh(\pi)}.$$

Infinite many solutions corresponding to different values of B_0

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Q3

Sep. of var., $u(x,y) \sim G(x)H(y)$

$$\Rightarrow -\left[\frac{G''-5G'+4G}{G}\right] = \boxed{\begin{array}{l}\dfrac{\ddot{H}}{H} = c \\ \dot{H}(0)=0 \\ \dot{H}(\pi)=0\end{array}} \Rightarrow \begin{array}{l} c = 0,\ -n^2 \quad n=1,2,3,\ldots \\ H_0(y) = 1 \\ H_n(y) = \cos(ny)\end{array}$$

From b.c. (i) and (ii), only $n=0, 2$ matter.

$n=0$: $\quad G_0'' - 5G_0' + 4G_0 = 0 \qquad$ Let $G_0(x) \sim e^{\alpha x}$

$\Rightarrow \alpha^2 - 5\alpha + 4 = 0 \qquad \alpha = 1, 4 \qquad G_0(x) = A_0 e^{x} + B_0 e^{4x}$

$n=2$: $\quad G_2'' - 5G_2' + 4G_2 = 4G_2$

$\Rightarrow G_2'' - 5G_2' = 0 \quad$ Let $G_2(x) \sim e^{\alpha x} \Rightarrow \alpha^2 - 5\alpha = 0, \ \alpha = 0, 5$

$\Rightarrow G_2(x) = A_2 + B_2 e^{5x}$

Full solution: $u(x,y) = (A_0 e^{x} + B_0 e^{4x}) + (A_2 + B_2 e^{5x})\cos(2y)$

From b.c. (i): $(A_0 + B_0) + (A_2 + B_2)\cos(2y) = 1 + \cos(2y)$

$\Rightarrow A_0 + B_0 = 1$ — ① $\qquad A_2 + B_2 = 1$ — ②

From b.c. (ii): $(A_0 e + B_0 e^4) + (A_2 + B_2 e^5)\cos(2y) = e^4 + e^5 \cos(2y)$

$\Rightarrow A_0 e + B_0 e^4 = e^4$ — ③ $\qquad A_2 + B_2 e^5 = e^5$ — ④

Solve ① & ③ $\Rightarrow A_0 = 0, \ B_0 = 1$

solve ② & ④ $\Rightarrow A_2 = 0, \ B_2 = 1$

Full solution: $u(x,y) = e^{4x} + e^{5x}\cos(2y)$

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Q4

Sep. of var., $u(x,y) \sim G(x)H(y)$

$$\Rightarrow \quad -\left(\frac{\ddot{H}}{H}+1\right) = \boxed{\frac{G''}{G} = c \quad \begin{array}{l} G'(0)=0 \\ G'(\pi)=0 \end{array}} \Rightarrow \begin{array}{l} c = 0, \ -n^2 \quad n=1,2,3\ldots \\ G_0(x)=1 \quad G_n(x)=\cos(nx) \end{array}$$

From b.c. (iii) and (iv), only $n = 0, 1, 2$ matter.

$n=0$: $\quad -\left(\frac{\ddot{H}_0}{H_0}+1\right)=0 \Rightarrow \ddot{H}_0 = -H_0 \Rightarrow H_0(y)=A_0\cos(y)+B_0\sin(y)$

$n=1$: $\quad -\left(\frac{\ddot{H}_1}{H_1}+1\right)=-1 \Rightarrow \ddot{H}_1 = 0 \Rightarrow H_1(y)=A_1 y + B_1$

$n=2$: $\quad -\left(\frac{\ddot{H}_2}{H_2}+1\right)=-4 \Rightarrow \ddot{H}_2 = 3H_2 \Rightarrow H_2(y)=A_2\cosh(\sqrt{3}y)+B_2\sinh(\sqrt{3}y)$

Full solution: $u(x,y)=A_0\cos(y)+B_0\sin(y) + (A_1 y + B_1)\cos(x)$
$\qquad + \left[A_2\cosh(\sqrt{3}y)+B_2\sinh(\sqrt{3}y)\right]\cos(2x).$

From b.c. (iii): $A_0 + B_1\cos(x) + A_2\cos(2x) = \cos(2x)$

$\Rightarrow A_0 = 0, \quad B_1 = 1, \quad A_2 = 0$

From b.c. (iv): $B_0\sin(1) + (A_1+1)\cos(x) + B_2\sinh(\sqrt{3})\cos(2x)$

$= 1 + 2\cos(x) + \cos(2x)$

$\Rightarrow B_0 = 1/\sin(1), \quad A_1 = 1, \quad B_2 = 1/\sinh(\sqrt{3})$

Full solution:

$$u(x,y) = \frac{\sin(y)}{\sin(1)} + (y+1)\cos(x) + \frac{\sinh(\sqrt{3}y)}{\sinh(\sqrt{3})}\cos(2x)$$

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