Q3

By Fourier series expansion

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx}$$

From the b.c. :

$C_0(0) = 1$, $C_1(0) = \frac{1}{2i}$, $C_2(0) = \frac{1}{2}$

All other $C_n(0) = 0$

$$\Rightarrow \dot{C}_n = \left[i(n^5 - n^3) - n^2 t + 4t \right] C_n$$

Only $n = 0, 1, 2$ (and $-1, -2$) matter.

$$\dot{C}_0 = 4t\, C_0 \Rightarrow C_0(t) = C_0(0)\, e^{2t^2} = e^{2t^2}$$

$$\dot{C}_1 = 3t\, C_1 \Rightarrow C_1(t) = C_1(0)\, e^{\frac{3}{2}t^2} = \frac{1}{2i} e^{\frac{3}{2}t^2}$$

$$\dot{C}_2 = i24\, C_2 \Rightarrow C_2(t) = C_2(0)\, e^{i24t} = \frac{1}{2} e^{i24t}$$

Full solution: $u(x,t) = C_0(t) + \left\{ \left[C_1(t) e^{ix} + C_2(t) e^{i2x} \right] + c.c. \right\}$

$$= e^{2t^2} + \left\{ \left[\left(\frac{-1}{2}\right) e^{\frac{3}{2}t^2} i e^{ix} + \frac{1}{2} e^{i(2x+24t)} \right] + c.c. \right\}$$

$$= e^{2t^2} + e^{\frac{3}{2}t^2} \sin(x) + \cos(2x + 24t)$$

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Q4

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$

From b.c. (i)

$C_2(0) = \frac{1}{2}$ $\quad (C_0(0) = 0)$

From b.c. (ii)

$\dot{C}_0(0) = 1$ $\quad (\dot{C}_2(0) = 0)$

$$\Rightarrow \ddot{C}_n = -n^2 C_n + in\dot{C}_n - 3\dot{C}_n + 4C_n$$

* only $n = 0, 2$ (and -2) matter

$n=0$: $\quad \ddot{C}_0 = -3\dot{C}_0 + 4C_0$

Let $C_0 \sim e^{\alpha t} \Rightarrow \alpha^2 + 3\alpha - 4 = 0$, $\alpha = 1, -4$

$\Rightarrow C_0(t) = A_0 e^{t} + B_0 e^{-4t}$

$$\left.\begin{aligned} C_0(0) = 0 &\Rightarrow A_0 + B_0 = 0 \\ \dot{C}_0(0) = 1 &\Rightarrow A_0 - 4B_0 = 1 \end{aligned}\right\} \Rightarrow A_0 = \frac{1}{5},\ B_0 = -\frac{1}{5}$$

$n=2$: $\quad \ddot{C}_2 = (i2-3)\dot{C}_2$

Let $D(t) \equiv \dot{C}_2(t) \Rightarrow \dot{D} = (i2-3)D \Rightarrow D(t) = D(0)e^{(i2-3)t}$

$\Rightarrow D(0) = \dot{C}_2(0) = 0 \Longrightarrow D(t) = 0$

$\dot{C}_2 = 0$

$\Rightarrow C_2(t) = C_2(0) = \frac{1}{2}$

Full solution: $u(x,t) = C_0(t) + \left[C_2(t)e^{i2x} + c.c.\right]$

$$= \frac{1}{5}e^{t} - \frac{1}{5}e^{-4t} + \cos(2x)$$

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