Q1

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(nx) \Rightarrow$$ From b.c. (iii), $a_0(0)=0$, $a_1(0)=1$, all other $a_n(0)=0$.

$$\underbrace{Q(x,t)}_{= t^2 e^{-t}\cos(x) + \cos(t) + 1} = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\cos(nx)$$

$\Rightarrow$ $q_0(t) = \cos(t)+1$, $q_1(t) = t^2 e^{-t}$, all other $q_n(t) = 0$

Only $n=0,1$ matter.

$$\frac{da_0}{dt} = q_0(t) = \cos(t)+1 \Rightarrow a_0(t) = \overset{0}{\overset{\parallel}{a_0(0)}} + \int_0^t (\cos(t)+1)\,dt$$
$$= \sin(t) + t$$

$$\frac{da_1}{dt} = -a_1 + q_1(t) \Rightarrow a_1(t) = a_1(0)e^{-t} + \int_0^t q_1(\hat{t})e^{-(t-\hat{t})}\,d\hat{t}$$
$$= e^{-t} + e^{-t}\int_0^t q_1(\hat{t})e^{\hat{t}}\,d\hat{t}$$
$$= e^{-t} + e^{-t}\int_0^t \hat{t}^2 \underbrace{e^{-\hat{t}}e^{\hat{t}}}_{=1}\,d\hat{t}$$
$$= e^{-t}\left(1 + \frac{t^3}{3}\right)$$

Full solution:

$$u(x,t) = a_0(t) + a_1(t)\cos(x)$$
$$= \sin(t) + t + e^{-t}\left(1 + \frac{t^3}{3}\right)\cos(x)$$

#

Q2

First, seek steady solution $u_s(x)$ which satisfies

$$\boxed{u_s'' = -2\sin(x), \quad u_s(0)=1, \quad u_s'(\pi) = -2}$$

$$u_s'(x) = A - 2\int \sin(x)\,dx = A + 2\cos(x)$$

$$u_s(x) = Ax + 2\int \cos(x)\,dx + B$$

$$= Ax + B + 2\sin(x)$$

From $u_s(0)=1 \Rightarrow A\cdot 0 + B + 2\sin(0) = 1 \Rightarrow B = 1$

From $u_s'(\pi) = -2 \Rightarrow A + 2\cos(\pi) = -2 \Rightarrow A = 0$

$$\Rightarrow u_s(x) = 1 + 2\sin(x)$$

Let $v(x,t) \equiv u(x,t) - u_s(x)$, then we have

$$\boxed{\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad v(0,t)=0, \quad v_x(\pi, t) = 0, \quad v(x,0) = \sin(x/2)}$$

It can be readily solved to obtain

$$v(x,t) = e^{-\frac{1}{4}t}\sin(x/2)$$

$\Rightarrow$ Full solution is $u(x,t) = 1 + 2\sin(x) + e^{-\frac{1}{4}t}\sin(x/2)$

#

Q3

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx} \Longrightarrow$$ From b.c. (i), $C_1(0) = \frac{1}{2i}$, $C_3(0) = \frac{1}{2i}$.

(also, $C_{-1}(0) = \frac{-1}{2i}$, $C_{-3}(0) = \frac{-1}{2i}$, but no need to process them)

all other $C_n(0) = 0$.

$$\underset{\displaystyle \cos(x)\cos(t)}{\underset{\displaystyle \parallel}{Q(x,t)}} = \sum_{n=-\infty}^{\infty} q_n(t) e^{inx}$$

$\searrow$ $q_1(t) = \frac{1}{2}\cos(t)$, (also, $q_{-1}(t) = \frac{1}{2}\cos(t)$, but no need to process it)

all other $q_n(t) = 0$

$\Rightarrow$ Only $n = 1, 3$ (and $-1, -3$) matter.

In general, $\dot{C}_n = (-n^2 + n^4) C_n + q_n(t)$.

For $n=1$, $\dot{C}_1 = q_1(t) = \frac{1}{2}\cos(t) \Rightarrow C_1(t) = C_1(0) + \frac{1}{2}\int_0^t \cos(t)\,dt$

$$= \frac{1}{2i} + \frac{1}{2}\sin(t)$$

For $n=3$, $\dot{C}_3 = 72\, C_3 \Rightarrow C_3(t) = C_3(0) e^{72t}$

$$= \frac{1}{2i} e^{72t}$$

Full solution:

$$u(x,t) = \left\{ C_1(t) e^{ix} + C_3(t) e^{i3x} \right\} + \text{c.c.}$$

$$= \left\{ \left(\frac{1}{2}\sin(t) - \frac{1}{2}i \right) e^{ix} + \left(\frac{-1}{2} e^{72t} \right) i\, e^{i3x} \right\} + \text{c.c.}$$

$$= \sin(t)\cos(x) + \sin(x) + e^{72t}\sin(3x)$$

#

Q4

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x) \Rightarrow \text{From b.c. (iii), (iv),}$$

$$a_1(0) = 1, \quad \dot{a}_1(0) = 1,$$

all other $a_n(0) = 0$, $\dot{a}_n(0) = 0$

$$\underbrace{Q(x,t)}_{\pi^2 \sin(\pi x)} = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x)$$

$$\searrow \quad \underbrace{q_1(t) = \pi^2, \text{ all other } q_n(t) = 0}_{\text{only } n=1 \text{ matters}}$$

In general,

$$\frac{d^2 a_n}{dt^2} = -(n\pi)^2 a_n + q_n$$

For $n=1$,

$$\frac{d^2 a_1}{dt^2} = -\pi^2 a_1 + \pi^2$$

Let $b \equiv a_1 - 1$

$$\Rightarrow \frac{d^2 b}{dt^2} = -\pi^2 b$$

$$\Rightarrow b(t) = A\cos(\pi t) + B\sin(\pi t)$$

$$\Rightarrow a_1(t) = A\cos(\pi t) + B\sin(\pi t) + 1$$

$$\Rightarrow \dot{a}_1(t) = -\pi A\sin(\pi t) + \pi B\cos(\pi t)$$

From $a_1(0) = 1 \Rightarrow A + 1 = 1 \Rightarrow A = 0$

From $\dot{a}_1(0) = 1 \Rightarrow \pi B = 1 \Rightarrow B = 1/\pi$

$$\Rightarrow a_1(t) = \frac{1}{\pi}\sin(\pi t) + 1$$

Full solution:

$$u(x,t) = a_1(t)\sin(\pi x)$$

$$= \left(\frac{1}{\pi}\sin(\pi t) + 1\right)\sin(\pi x)$$

#