Q1

by MOC,

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{tx}{1+t} \\[2ex] \dfrac{du}{dt} = \dfrac{tu}{1+t} \end{cases}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{t}{1+t}\,dt = \left(1 - \frac{1}{1+t}\right)dt$$

$$\int_{t=0}^{t=t} \frac{dx}{x} = \int_{t=0}^{t=t} \left(1 - \frac{1}{1+t}\right)dt$$

$$\Rightarrow \ln\left(\frac{x(t)}{x(0)}\right) = t - \ln(1+t)$$

$$\Rightarrow \ x(t) = x(0)\cdot \frac{e^t}{1+t} \quad \text{— ①}$$

similarly

$$u(t) = u(0)\,\frac{e^t}{1+t} \quad \text{— ②}$$

From the b.c., $u(0) = e^{-[x(0)]^2}$

From ①, $x(0) = x(t)\,(1+t)\,e^{-t}$

$$\Rightarrow \ u(t) = e^{-[x(t)(1+t)e^{-t}]^2} \cdot \left(\frac{e^t}{1+t}\right)$$

Full solution:

$$u(x,t) = e^{-[x(1+t)e^{-t}]^2} \cdot \left(\frac{e^t}{1+t}\right)$$

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Q2

By MOC $\quad \dfrac{dx}{dt} = 1+u = 1 + u(0)e^t \Rightarrow x(t) = x(0) + t + u(0)(e^t - 1)$ — 1

$\dfrac{du}{dt} = u \Rightarrow u(t) = u(0)e^t$ — 2

- when $x(0) < 0$, $x(t) = x(0) + t$, $u(0) = 0$

$x(t) < t \qquad u(t) = 0$

$$\boxed{u(x,t) = 0 \text{ when } x < t}$$

- when $0 \le x(0) \le 1$, $x(t) = x(0) + t + 2x(0)(e^t - 1)$, $u(0) = 2x(0)$

$x(0) = \dfrac{x(t) - t}{2e^t - 1} \qquad u(t) = 2x(0)e^t$

$t \le x(t) \le 2e^t - 1 + t$

$u(t) = 2e^t \cdot \left(\dfrac{x(t) - t}{2e^t - 1}\right)$

$$\boxed{u(x,t) = 2e^t\left(\frac{x-t}{2e^t-1}\right) \text{ when } t \le x \le 2e^t - 1 + t}$$

- when $x(0) > 1$, $x(t) = x(0) + t + 2(e^t - 1)$, $u(0) = 2$

$x(t) > 2e^t - 1 + t \qquad u(t) = 2e^t$

$$\boxed{u(x,t) = 2e^t \text{ when } x > 2e^t - 1 + t}$$

Q3

by MOC,

$$\begin{cases} \dfrac{dx}{dt} = 2e^{u} = 2\left(e^{u(0)} + t\right) \Rightarrow x(t) = x(0) + 2t\,e^{u(0)} + t^2 \quad ① \\[2ex] \dfrac{du}{dt} = e^{-u} \Rightarrow e^{u}\,du = dt \end{cases}$$

$$\Rightarrow e^{u(t)} - e^{u(0)} = t$$

$$\Rightarrow u(t) = \ln\left(e^{u(0)} + t\right) \quad ②$$

when $x(0) < 0$, $u(0) = 0 \Longrightarrow \underline{\underline{u(t) = \ln(1+t)}}$

$$\Downarrow$$

$$x(t) = x(0) + 2t + t^2$$

$$\underline{\underline{x(t) < 2t + t^2}}$$

$$\boxed{u(x,t) = \ln(1+t) \quad \text{when} \quad x < 2t + t^2}$$

when $x(0) \geq 0$, $u(0) = \ln(x(0)+1) \Rightarrow \underline{\underline{u(t) = \ln(x(0)+1+t)}}$

$$\Downarrow$$

$$x(t) = x(0) + 2t\,[x(0)+1] + t^2$$

$$\underline{\underline{x(t) \geq 2t + t^2}}$$

$$\Downarrow$$

$$x(0) = \frac{x(t) - 2t - t^2}{1+2t}$$

$$\downarrow$$

$$u(t) = \ln\left[\frac{x(t) - 2t - t^2}{1+2t} + 1 + t\right]$$

$$\boxed{u(x,t) = \ln\left[\left(\frac{x - 2t - t^2}{1+2t}\right) + 1 + t\right] \quad \text{when} \quad x \geq 2t + t^2}$$

Q4

The PDE can be written as

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u}_{w} + \underbrace{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u}_{w} = 0$$

$$\Downarrow$$

$$w(x,t) = u_t(x,t) + u_x(x,t)$$

$$\Downarrow$$

$$w(x,0) = u_t(x,0) + u_x(x,0) = x + 1 \quad \text{— (iii)}$$

$$\Rightarrow \begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial w}{\partial x} = -w & \text{— ①} \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = w & \text{— ②} \end{cases}$$

Solving ① with b.c. (iii):

$$\frac{dx}{dt} = -1 \Rightarrow x(t) = x(0) - t \Rightarrow x(0) = x(t) + t$$

$$\frac{dw}{dt} = -w \Rightarrow w(t) = w(0)e^{-t} = (x(0)+1)e^{-t} = (x(t)+t+1)e^{-t}$$

$$\Rightarrow \quad w(x,t) = (x+t+1)e^{-t}$$

Solving ② (with $w(x,t)$ now known) with b.c. (i), :

$$\frac{dx}{dt} = 1 \Rightarrow x(t) = x(0) + t \Rightarrow x(0) = x(t) - t$$

$$\frac{du}{dt} = (x+t+1)e^{-t} = (x(0)+2t+1)e^{-t}$$

$$\Rightarrow u(t) = u(0) + (x(0)+1)(1-e^{-t}) + 2 - e^{-t}(2+2t)$$

$$= x(0) + (x(0)+1)(1-e^{-t}) + 2 - e^{-t}(2+2t)$$

$$= (x(t)-t) + (x(t)-t+1)(1-e^{-t}) + 2 - e^{-t}(2+2t)$$

$\Rightarrow$ Full Solution:

$$u(x,t) = x - t + (x-t+1)(1-e^{-t}) + 2 - e^{-t}(2+2t)$$

$$= \underbrace{[2-e^{-t}]}_{A(t)}x + \underbrace{[3-2t-(3+t)e^{-t}]}_{B(t)}$$

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