PLEASE READ THIS PAGE FIRST

A statement on collaboration is required for all reports, including those that are done independently without any collaboration. See instruction below on how to prepare the statement. Please read the rules before forming any collaboration for the homework. A violation of the rule(s) given in this page will be considered a violation of ASU's Academic Integrity Policy.

Rules on collaboration for homework:

(1) Collaboration is not allowed unless all involved follow rules (2)-(3) and unless the extent of collaboration is properly disclosed in a statement in the first page of the report for the assignment. See additional instruction below for the required content of the statement.

(2) For each assignment, each person can have maximum of one collaborator. Be aware that a collaborator's collaborator counts as a collaborator. For example, if Alice collaborates with Bob and Bob collaborates with Charles, Charles counts as a collaborator of Alice. All three violate the rule. In other words, collaboration can only be carried out within an isolated "team of two". Please talk to a potential collaborator to ensure that this rule is not violated before establishing any collaboration.

(3) In a legitimate collaboration, each individual must make a non-negligible contribution to the collaborative effort. Taking the solution or code from another student without making a contribution to it is not allowed. To certify that a collaboration is legitimate, the submitter's contribution to the collaborative effort must be documented in the statement on collaboration.

The statement on collaboration

This statement is mandatory and must be placed in the beginning of the first page of report. If no collaboration occurred, simply state "**No collaboration**". This implies that the person submitting the report ***has not helped anyone or received help from anyone*** in the process of completing the assignment. If collaboration occurred, provide the name of collaborator (only one allowed), a list of the task(s) on which collaboration occurred, and descriptions of the contribution by the submitter to the collaborative effort. Example:

Name of collaborator: Joe Smith	
Task	Contribution to collaborative effort
Problem 1	Developed matlab code with collaborator
Problem 3	Compared mathematical derivation with collaborator

MAE/MSE 502, Spring 2021 Homework #1 (1 point ≈ 1% of the total score for the semester.)

Please upload your work to Canvas as a single PDF file. Please follow the rules on collaboration as described in the preceding page. A statement on collaboration is required for all reports, including those that are produced independently. If there is no collaboration, write "No collaboration". Please include the printout of computer code(s) (if any) in the report.

Problem 1 (4 points)
(a) For $u(x, t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions (u_x denotes $\partial u/\partial x$),

(i) $u_x(0, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x, 0) = P(x)$,

where

$$P(x) = x^5 - 3x^3 + 2x^2 + [\sin(\pi x)]^8 \ .$$

Plot the solution, $u(x,t)$, as a function of x at t = 0, 0.005, 0.02, and 0.1. Please collect all four curves in a single plot.

(b) Repeat (a) but now solve the system with the first two boundary conditions given as

(i) $u_x(0, t) = 0$, (ii) $u(1, t) = 0$.

(The 3rd boundary condition remains unchanged.) Plot the solution, $u(x,t)$, as a function of x at t = 0, 0.005, 0.02, and 0.1. Please collect all four curves in a single plot.

(c) Repeat (a) but now solve the system with the first two boundary conditions given as

(i) $u(0, t) = 0$, (ii) $u(1, t) = 0$

(The 3rd boundary condition remains unchanged.) Plot the solution, $u(x,t)$, as a function of x at t = 0, 0.005, 0.02, and 0.1. Please collect all four curves in a single plot.

Note for Problem 1: (i) We expect the solution to be expressed as an infinite series which needs to be truncated to allow numerical computations. It is your job to determine the appropriate number of terms to keep. Note that the solution at t = 0 should match the given initial state in the 3rd boundary condition. If they do not match, either the solution is wrong or more terms need to be retained in the series. (ii) To evaluate the expansion coefficients in the infinite series, there is no need to perform the integrations analytically. (If you wish to do so, please feel free to use an online integrator or similar software such as Wolfram Alpha. No need to do it by hand.) It is perfectly acceptable (actually, recommended for this problem) to evaluate the integrals numerically using Matlab or its equivalent. See Additional Note in the last page for an example of using Matlab to numerically evaluate an integral.

Problem 2 (1.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the PDE,

$$(1+t)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \pi^2 u \ ,$$

with the boundary conditions,

(i) $u_x(0, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x,0) = \cos(\pi x) + \cos(2\pi x)$.

In addition to the full solution, please also find the steady solution. Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 0.01, 0.03, 0.07, and as $t \to \infty$ (i.e., the steady solution). Collect all five curves in one plot. For this problem, we expect a closed-form analytic solution with only a finite number of terms and without any unevaluated integral. This requirement applies to both the full solution and the steady solution. Expect a deduction if the solution is written as an infinite series or if it contains any unevaluated integrals.

Problem 3 (1.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le \pi$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = 0.01\ \frac{\partial^2 u}{\partial x^2} + u \ ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(\pi, t) = 0$, and (iii) $u(x,0) = \sin(5x) + \sin(10x) + \sin(20x)$

For this problem, we expect a closed-form analytic solution with only a finite number of terms and without any unevaluated integral. Expect a deduction if the solution is written as an infinite series or if it contains any unevaluated integrals.

Problem 4 (1 point)
Consider the ODE for $u(x)$ defined over the interval, $p \le x \le q$,

$$\frac{d^2 u}{dx^2} = 1 \ ,$$

with boundary conditions

$$u'(p) = P \quad , \ u'(q) = Q \ ,$$

where u' denotes du/dx. As mentioned in Lecture 3, the existence and uniqueness of the solution(s) for the system generally depend on the values of (p, q, P, Q).

(i) Find an example of the values of (p, q, P, Q) for which the system has a unique solution. If you determine that it is not possible for the system to have a unique solution, please explain why.
(ii) Find an example of the values of (p, q, P, Q) for which the system has no solution at all. If you determine that the system always has a solution (or solutions), please explain why.
(iii) Find an example of the values of (p, q, P, Q) for which the system has multiple solutions. (In other words, the solutions for the system exist but are not unique.) If you determine that it is not possible for the system to have multiple solutions, please explain why.

Problem 5 (1 point)
For the heat transfer problem, the Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K\frac{\partial^2 u}{\partial \hat{x}^2}\,,\quad 0 \le \hat{x} \le L \quad (L \text{ is the length of the "metal rod", in meters) and } \hat{t} \ge 0\,, \quad (1)$$

where $\hat{t}$ and $\hat{x}$ are time in seconds and distance in meters, K is thermal diffusivity in m^2/s, and u is temperature. In our class, we usually consider the non-dimensionalized version of (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,,\quad 0 \le x \le 1 \;\text{ and }\; t \ge 0\;, \qquad (2)$$

where x is related to $\hat{x}$ by $\hat{x} = L\,x$. In (2), time is also non-dimensionalized by $\hat{t} = T\,t$, where T is a certain dimensional time scale, and so on.

(a) In order to claim that the non-dimensionalized system, (1), is equivalent to its dimensional counterpart, (2), the three parameters K, L, and T must satisfy a unique relation. First, find out what this relation is. (For the discussion in part (b) & (c), it is useful to write the relation as $T = f(K, L)$.)

(b) Suppose that the non-dimensionalized Heat equation, (2), is used to model the real world problem of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(c) Same as (b), but suppose that (2) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place in the scenario described in Part (d).)

(d) Are the time scales you obtained in (b) and (c) consistent with daily experience? For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the spoon is made entirely of copper. Note that the time for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.

Additional Note: Using Matlab to numerically evaluate an integral

One of the simplest Matlab functions for this purpose is **trapz**. It uses the trapezoidal method to evaluate an integral. For example, to numerically evaluate

$$I = \int_0^1 \sin(x)dx \ ,$$

with $\Delta x = 0.01$, we first construct the discretized arrays of the coordinate points and the values of the integrand at those points. We then call **trapz** with those two arrays as the input to complete the integration. This takes just 3 lines of Matlab code:

```
x = [0:0.01:1];
y = sin(x);
Integ = trapz(x,y)
```

The outcome can be readily verified with the analytic result of $I = 1 - \cos(1)$.

When using **trapz** or similar functions for numerical integration, it is important to support it with a sufficient grid resolution to avoid "aliasing errors". It requires a minimum of 5 points (10 points are recommended) to resolve one cycle of a sinusoidal wave. In the example above, 100 grid points are used to resolve less than 1/6 of a cycle, which is sufficient. If the integrand in the above example is changed to $y = \sin(500\,x)$, the result of numerical integration will be entirely wrong. The remedy would be to also increase the grid resolution. For example, change the grid to x = [0:0.0001:1].