Q2

separation of var. : $u \sim G(x)H(t)$

b.c. (i),(iii) $G'(0)=0,\ G'(1)=0$

$$\Rightarrow (1+t)\,G\dot{H} = G''H + \pi^2 GH \;\Rightarrow\; (1+t)\frac{\dot{H}}{H} - \pi^2 = \frac{G''}{G} = c$$

$c = 0,\ -(n\pi)^2$
$n = 1, 2, 3, \dots$

$G_0(x) = 1, \quad G_n(x) = \cos(n\pi x)$

$c=0 \quad (1+t)\dfrac{\dot{H}_0}{H_0} = \pi^2$

$c\neq 0 \quad (1+t)\dfrac{\dot{H}_n}{H_n} = \pi^2 - (n\pi)^2$

observing b.c. (iii), only $\underline{n = 1, 2}$ matter.

$n=1 \Rightarrow \dot{H}_1 = 0 \quad H_1(t) = \text{const (set to 1)}$

$$n=2 \Rightarrow (1+t)\frac{\dot{H}_2}{H_2} = \pi^2 - 4\pi^2 = -3\pi^2 \Rightarrow H_2(t) = \overbrace{H_2(0)}^{\text{OK to set to 1}}(1+t)^{-3\pi^2}$$

Full solution:
$$u(x,t) = a_1 G_1(x) H_1(t) + a_2 G_2(x) H_2(t)$$
$$= a_1 \cos(\pi x) + a_2 (1+t)^{-3\pi^2}\cos(2\pi x)$$

From b.c. (iii), $a_1 = 1,\ a_2 = 1$.

$$\Rightarrow \underline{u(x,t) = \cos(\pi x) + \cos(2\pi x)(1+t)^{-3\pi^2}} \quad \#$$

As $t \to \infty$, $(1+t)^{-3\pi^2} \to 0$.

steady solution is $\underline{u_s(x) = \cos(\pi x)}$ #

Q3

seperation of var, $u \sim G(x)H(t)$

$$\Rightarrow \quad G\dot{H} = 0.01\, G''H + GH \Rightarrow 100\left(\frac{\dot{H}}{H} - 1\right) = \boxed{\frac{G''}{G} = c \quad G(0)=0 \quad G(\pi)=0}$$

b.c. (i), (ii)

$$100\left(\frac{\dot{H}_n}{H_n} - 1\right) = c_n = -n^2$$

$$c = -n^2, \quad n = 1, 2, 3, \ldots$$

$$G_n(x) = \sin(nx)$$

Observing b.c. (iii), only $n = 5$, 10, and 20 are relevant.

$$n=5: \quad \dot{H}_5 = \left(\frac{3}{4}\right) H_5 \Rightarrow H_5(t) = \underbrace{H_5(0)}_{\text{set to } 1} e^{\frac{3}{4}t}$$

$$n=10: \quad \dot{H}_{10} = 0 \Rightarrow H_{10}(t) = \text{const} \leftarrow \text{set to } 1$$

$$n=20: \quad \dot{H}_{20} = -3H_{20} \Rightarrow H_{20}(t) = \underbrace{H_{20}(0)}_{\text{set to } 1.} e^{-3t}$$

Full solution:

$$u(x,t) = a_5\, G_5(x) H_5(t) + a_{10}\, G_{10}(x) H_{10}(t) + a_{20}\, G_{20}(x) H_{20}(t)$$

$$= a_5 \sin(5x) e^{\frac{3}{4}t} + a_{10} \sin(10x) + a_{20} \sin(20x) e^{-3t}$$

From b.c. (iii), $a_5 = 1$, $a_{10} = 1$, $a_{20} = 1$.

$$\Rightarrow \quad \underline{u(x,t) = \sin(5x) e^{\frac{3}{4}t} + \sin(10x) + \sin(20x) e^{-3t}}$$

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