Q2b

separation of var. $u \sim G(x)H(y)$

$$\Rightarrow \quad -\frac{\ddot{H}}{H} = \boxed{\frac{G''}{G} = c \qquad \begin{matrix} G'(0)=0 \\ G'(1)=0 \end{matrix}}$$

b.c. (i) & (ii)

$$\Rightarrow \quad c = 0, \; -(n\pi)^2, \; n = 1, 2, 3, \ldots$$

$$G_0(x) = 1, \quad G_n(x) = \cos(n\pi x)$$

$c=0$: $\ddot{H}_0 = 0$

$$\Rightarrow H_0(y) = A_0 y + B_0$$

$c = -(100\pi)^2$: $\ddot{H}_{100} = +(100\pi)^2 H_{100} \Rightarrow H_{100}(y) = A_{100}\cosh(100\pi y) + B_{100}\sinh(100\pi y)$

Full solution:

$$u(x,y) = A_0 y + B_0 + \left[A_{100}\cosh(100\pi y) + B_{100}\sinh(100\pi y)\right]\cos(100\pi x)$$

$$\Rightarrow u_y(x,y) = A_0 + 100\pi\left[A_{100}\sinh(100\pi y) + B_{100}\cosh(100\pi y)\right]\cos(100\pi x)$$

b.c. (iii): $\quad 100 = A_0 + 100\pi B_{100}\cos(100\pi x) \quad \Rightarrow A_0 = 100, \; B_{100} = 0.$

b.c. (iv): $\quad 100 + \cos(100\pi x) = 100 + 100\pi A_{100}\sinh(100\pi)\cos(100\pi x)$

$$\Rightarrow \quad A_{100} = \frac{1}{100\pi\sinh(100\pi)}$$

At the end, B_0 remains undetermined.

Full solution:

$$u(x,y) = B_0 + 100y + \frac{\cosh(100\pi y)\cos(100\pi x)}{100\pi\sinh(100\pi)} \quad \#$$

Infinite many solutions, associated with different values of B_0.

Q3

separation of var., $u \sim G(x)H(y)$

$$\Rightarrow \quad -\frac{\ddot{H}}{H} - 16 = \boxed{\frac{G''}{G} = c \qquad \begin{matrix} G'(0)=0 \\ G'(\pi)=0 \end{matrix}} \longleftarrow \text{b.c. (i) \& (ii)}$$

$c = 0,\ -n^2 \quad n = 1, 2, 3, \ldots$

$G_0(x) = 1, \quad G_n(x) = \cos(nx)$

$$\ddot{H}_n = (n^2 - 16) H_n$$

observing b.c. (iii), (iv), only $n = 0, 4, 5$ matter.

$n = 0: \quad \ddot{H}_0 = -16 H_0 \Rightarrow H_0(y) = A_0 \cos(4y) + B_0 \sin(4y)$

$n = 4: \quad \ddot{H}_4 = 0 \Rightarrow H_4(y) = A_4 y + B_4$

$n = 5: \quad \ddot{H}_5 = 9 H_5 \Rightarrow H_5(y) = A_5 \cosh(3y) + B_5 \sinh(3y)$

Full solution:

$$u(x,y) = A_0 \cos(4y) + B_0 \sin(4y) + (A_4 y + B_4)\cos(4x) + \big(A_5 \cosh(3y) + B_5 \sinh(3y)\big)\cos(5x)$$

b.c. (iii): $\quad 1 + \cos(4x) = A_0 + B_4 \cos(4x) + A_5 \cos(5x)$

$$\Rightarrow A_0 = 1, \quad B_4 = 1, \quad A_5 = 0$$

b.c. (iv): $\quad \cos(5x) = \cos(4) + B_0 \sin(4) + (A_4 + 1)\cos(4x) + B_5 \sinh(3)\cos(5x)$

$\Rightarrow \cos(4) + B_0 \sin(4) = 0 \Rightarrow B_0 = -\cot(4) \qquad A_4 + 1 = 0 \Rightarrow A_4 = -1$

$B_5 \sinh(3) = 1 \Rightarrow B_5 = 1/\sinh(3)$

Full solution:

$$u(x,y) = \cos(4y) - \cot(4)\sin(4y) + (1-y)\cos(4x) + \frac{\sinh(3y)\cos(5x)}{\sinh(3)}$$

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Q4

seperation of var. $u \sim G(x)H(y)$

$$\Rightarrow \quad \frac{-G''+2G'-G}{G} = \boxed{\frac{\ddot{H}}{H} = c \qquad \begin{matrix} H(0)=0 \\ H(\pi)=0 \end{matrix}} \leftarrow \text{b.c. (iii), (iv)}$$

$$\Downarrow \quad c = -n^2 \quad n = 1, 2, 3, \dots$$

$$H_n(y) = \sin(ny)$$

$$\downarrow$$

$$G_n'' - 2G_n' + (1-n^2)G_n = 0$$

Observing b.c. (i), (ii), only $n=2$ is relevant.

$$G_2'' - 2G_2' - 3G_2 = 0$$

Let $G_2 \sim e^{\alpha x} \Rightarrow \alpha^2 - 2\alpha - 3 = 0 \Rightarrow \alpha = 3, -1$

$$G_2(x) = A_2 e^{3x} + B_2 e^{-x}$$

Full solution: $u(x,y) = \left(A_2 e^{3x} + B_2 e^{-x}\right)\sin(2y)$

b.c. (i): $15\sin(2y) = (A_2 + B_2)\sin(2y)$

b.c. (ii): $0 = (A_2 e^{3\ln 2} + B_2 e^{-\ln 2})\sin(2y)$

$$= (8A_2 + \tfrac{1}{2}B_2)\sin(2y)$$

$$\Rightarrow \begin{cases} A_2 + B_2 = \cancel{0}\,15 \\ 8A_2 + \frac{1}{2}B_2 = 0 \end{cases} \Rightarrow A_2 = -1, \; B_2 = 16$$

Full solution:

$$u(x,y) = (16e^{-x} - e^{3x})\sin(2y)$$

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