MAE/MSE 502, Spring 2021, Homework #3 (12 points)

See Homework #1 for rules on collaboration.

Problem 1 (3 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, consider the 1-D Wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(1, t) = 0$, (iii) $u(x, 0) = P(x)$, (iv) $u_t(x, 0) = 0$.

(a) Solve the system with P(x) given as

$$\begin{aligned} P(x) &= 3\,x \quad &&, \text{ if } 0 \le x \le 0.25 \\ &= 1 - x &&, \text{ if } 0.25 < x \le 1 , \end{aligned}$$

and plot the solution as a function of x at t = 0, 0.3, 0.5, 0.7, 1.0, and 1.8. Please collect all 6 curves in one plot.

(b) Solve the system with $P(x)$ given as (this emulates a “wave packet”)

$$P(x) = \sum_{n=30}^{50} \exp\left[-\left(\frac{n-40}{4}\right)^2\right] \sin(n\pi x) ,$$

and plot the solution as a function of x at t = 0, 0.5, 1.0, and 1.5. For this part, it is recommended that the four curves be plotted separately, for example arranged in 4 panels using "subplot" in Matlab.

Problem 2 (3 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, consider the PDE (in which U, K, and B are constants)

$$\frac{\partial u}{\partial t} = U\frac{\partial u}{\partial x} + K\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^3 u}{\partial x^3} ,$$

with periodic boundary conditions in the x-direction (i.e., $u(0, t) = u(2\pi, t)$, $u_x(0, t) = u_x(2\pi, t)$, and so on), and the boundary condition in the t-direction given as

$$u(x, 0) = \frac{(1-\cos(x))^{10}}{1024} .$$

Solve the PDE by Fourier series expansion. Plot the solution $u(x, t)$ at $t = 0.1$ for the three cases with
(i) $U = 10$, $K = 0$, $B = 0$ (ii) $U = 10$, $K = 2.5$, $B = 0$, and (iii) $U = 0$, $K = 0$, $B = 0.25$. Also, plot the solution at $t = 0$ (which is the same for all three cases). Please collect all four curves in one plot.

Problem 3 (2 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$(1+t)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + 4\,u$$

with periodic boundary conditions in the x-direction, and the boundary condition in the t-direction given as

$$u(x, 0) = 1 + \sin(2x) + \cos(2x)\ .$$

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals. The solution should be expressed in real functions.

Problem 4 (2 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = t\,\frac{\partial^2 u}{\partial x^2} + t\,\frac{\partial^3 u}{\partial x^3} + (1+t)\frac{\partial^4 u}{\partial x^4} - u$$

with periodic boundary conditions in the x-direction, and the boundary condition in the t-direction given as

$$u(x,0) = 1 + \sin(x)\ .$$

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals. The solution should be expressed in real functions.

Problem 5 (2 points) For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\,\partial t} + 4\,u$$

with periodic boundary conditions in the x-direction, and the boundary conditions at $t = 0$ given as

(i) $u(x,0) = 1$

(ii) $u_t(x,0) = \cos(2x)$.

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals. The solution should be expressed in real functions.