# Q3

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx}$$

$\Rightarrow$

$$(1+t)\dot{C}_n = -n^2 C_n + in C_n + 4C_n$$

$n=0$: $\quad \dot{C}_0 = \frac{4}{1+t} C_0$

$\Rightarrow \quad C_0(t) = \underbrace{C_0(0)}_{=1} (1+t)^4 = (1+t)^4$

From b.c.:

$$\underbrace{u(x,0)}_{=\,1+\sin(2x)+\cos(2x)} = \sum_{n=-\infty}^{\infty} C_n(0)\, e^{inx}$$

$$1+\sin(2x)+\cos(2x) = 1 + \left[\left(\frac{1}{2i}+\frac{1}{2}\right)e^{i2x} + c.c.\right]$$

$\Rightarrow \quad C_0(0) = 1 \quad C_2(0) = \frac{1}{2i} + \frac{1}{2}$

only $n=0, \pm 2$ matter

$n=2$: $\quad \dot{C}_2 = \frac{i2}{1+t} C_2 \quad \Rightarrow \quad C_2(t) = C_2(0)(1+t)^{i2}$

$$= \left(\frac{1}{2i}+\frac{1}{2}\right)(1+t)^{i2}$$

$$= \left(\frac{1}{2}-\frac{i}{2}\right)e^{i2\ln(1+t)}$$

Full solution:

$$u(x,t) = C_0(t) + \left[C_2(t)e^{i2x} + c.c.\right]$$

$$= (1+t)^4 + \left[\underbrace{\left(\frac{1}{2}-\frac{i}{2}\right)e^{i2\ln(1+t)}\, e^{i2x}} + c.c.\right]$$

$$\left(\frac{1}{2}-\frac{i}{2}\right)e^{i(2x+2\ln(1+t))}$$

$$= \left(\frac{1}{2}-\frac{i}{2}\right)\left[\cos(2x+2\ln(1+t)) + i\sin(2x+2\ln(1+t))\right]$$

$$= \underbrace{\frac{1}{2}\left[\cos(2x+2\ln(1+t)) + \sin(2x+2\ln(1+t))\right]}_{\circledast \text{ real part}} + i\underbrace{\star}_{\text{imaginary part}}$$

$$= (1+t)^4 + 2\circledast$$

$$= (1+t)^4 + \cos(2x+2\ln(1+t)) + \sin(2x+2\ln(1+t))$$

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Q4

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx}$$

$$\Rightarrow \dot{C}_n = \left[-n^2 t - i n^3 t + (1+t) n^4 - 1\right] C_n$$

$n=0$: $\dot{C}_0 = -C_0 \Rightarrow C_0(t) = C_0(0) e^{-t} = e^{-t}$

$n=1$: $\dot{C}_1 = -itC_1 \Rightarrow C_1(t) = C_1(0) e^{-i\frac{t^2}{2}} = \frac{1}{2i} e^{-i\frac{t^2}{2}}$

From b.c.:

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}$$

$$\parallel$$

$$1 + \sin(x)$$

$$\parallel$$

$$1 + \left[\frac{1}{2i} e^{ix} + c.c.\right]$$

$\Rightarrow C_0(0) = 1$, $C_1(0) = \frac{1}{2i}$

only $n = 0, \pm 1$ matter

Full solution:

$$u(x,t) = C_0(t) + \left[C_1(t) e^{ix} + c.c.\right]$$

$$= e^{-t} + \Big[\underbrace{\frac{1}{2i} e^{-i\frac{t^2}{2}} e^{ix}} + c.c.\Big]$$

$$\parallel$$

$$\left(\frac{-i}{2}\right) e^{i(x - \frac{t^2}{2})}$$

$$\parallel$$

$$\left(\frac{-i}{2}\right)\left[\cos(x - \tfrac{t^2}{2}) + i \sin(x - \tfrac{t^2}{2})\right]$$

$$\parallel$$

$$\underbrace{\frac{1}{2}\sin(x - \tfrac{t^2}{2})}_{\circledast \text{ real part}} + i \underbrace{\ast}_{\text{~~real~~ imaginary part}}$$

$$= e^{-t} + 2 \cdot \circledast$$

$$= e^{-t} + \sin(x - \tfrac{t^2}{2})$$

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## Q5

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$

$$\Rightarrow \ddot{C}_n = -n^2 C_n + in\dot{C}_n + 4C_n$$

$n=0$: $\ddot{C}_0 = 4C_0$

$C_0(t) = A_0 \cosh(2t) + B_0 \sinh(2t)$

$\Rightarrow \dot{C}_0(t) = 2A_0 \sinh(2t) + 2B_0 \cosh(2t)$

$C_0(0) = 1 \Rightarrow A_0 = 1 \quad \dot{C}_0(0) = 0 \Rightarrow B_0 = 0$

$\Rightarrow C_0(t) = \cosh(2t)$

$n=2$: $\ddot{C}_2 = i2\dot{C}_2$

Let $D \equiv \dot{C}_2 \Rightarrow \dot{D} = i2D$

$\Rightarrow D(t) = D(0)e^{i2t} = \dot{C}_2(0)e^{i2t} = \frac{1}{2}e^{i2t}$

From b.c. (i)

$$\underbrace{u(x,0)}_{=1} = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}$$

$\Rightarrow C_0(0) = 1$, all other $C_n(0) = 0$

From b.c. (ii)

$$\underbrace{u_t(x,0)}_{=\cos(2x)} = \sum_{n=-\infty}^{\infty} \dot{C}_n(0) e^{inx}$$

$\cos(2x) = \frac{1}{2}e^{i2x} + \text{c.c.}$

$\Rightarrow \dot{C}_2(0) = \frac{1}{2}$, all other $\dot{C}_n(0) = 0$

Only $n = 0, \pm 2$ matter

$$\dot{C}_2 = D = \frac{1}{2}e^{i2t} \Rightarrow C_2(t) = \underbrace{C_2(0)}_{=0} + \int_0^t \frac{1}{2}e^{i2t}\,dt = \frac{1}{4i}[e^{i2t} - 1]$$

Full solution:

$$u(x,t) = C_0(t) + [C_2(t)e^{i2x} + \text{c.c.}]$$

$$= \cosh(2t) + \left\{\frac{1}{4i}[e^{i2t} - 1]\, e^{i2x} + \text{c.c.}\right\}$$

$$= \cosh(2t) + \left\{\frac{1}{4i}[e^{i(2x+2t)} - e^{i2x}] + \text{c.c.}\right\}$$

$$= \cosh(2t) + \frac{1}{2}\sin(2x+2t) - \frac{1}{2}\sin(2x)$$

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