Q1

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(n\pi x)$$

$$\underbrace{Q(x,t)}_{\substack{\shortparallel \\ e^{-\pi^2 t} + \cos(\pi x) e^{-2\pi^2 t}}} = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\cos(n\pi x) \longrightarrow \begin{array}{l} q_0(t) = e^{-\pi^2 t} \\ q_1(t) = e^{-2\pi^2 t} \end{array}$$

$$\underbrace{u(x,0)}_{\substack{\shortparallel \\ 1+\cos(\pi x)}} = a_0(0) + \sum_{n=1}^{\infty} a_n(0)\cos(n\pi x) \longrightarrow a_0(0) = 1,\ a_1(0) = 1$$

Only $n = 0, 1$ matter.

$$\dot{a}_0 = -\pi^2 a_0 + q_0 = -\pi^2 a_0 + e^{-\pi^2 t}$$

$$\Rightarrow a_0(t) = a_0(0) e^{-\pi^2 t} + \int_0^t e^{-\pi^2 \hat{t}} e^{-\pi^2 (t-\hat{t})} d\hat{t}$$

$$= (1+t) e^{-\pi^2 t}$$

$$\dot{a}_1 = -\pi^2 a_1 - \pi^2 a_1 + q_1 = -2\pi^2 a_1 + e^{-2\pi^2 t}$$

$$\Rightarrow a_1(t) = a_1(0) e^{-2\pi^2 t} + \int_0^t e^{-2\pi^2 \hat{t}} e^{-2\pi^2 (t-\hat{t})} d\hat{t}$$

$$= (1+t) e^{-2\pi^2 t}$$

Full solution:

$$u(x,t) = a_0(t) + a_1(t)\cos(\pi x)$$

$$= (1+t) e^{-\pi^2 t} + (1+t) e^{-2\pi^2 t}\cos(\pi x)$$

#

Q2

Stedy solution $u_s(x)$ satisfies

$$\begin{cases} u_s'' + \cos(x) = 0 \\ u_s(0)=0, \quad u_s(2\pi) = 2\pi \end{cases} \Rightarrow u_s(x) = \cos(x) + Ax + B$$

$\rightarrow A = 1, \quad B = -1$

$$\Rightarrow u_s(x) = \cos(x) + x - 1$$

Let $v(x,t) \equiv u(x,t) - u_s(x)$

$$\Rightarrow \begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} \\ v(0,t)=0, \quad v(2\pi,t)=0, \quad v(x,0) = \sin\left(\frac{x}{2}\right) \end{cases}$$

By separation of var., we have

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{nx}{2}\right) e^{-\left(\frac{n}{2}\right)^2 t}$$

From the last b.c., $a_1 = 1$ and all other $a_n = 0$.

$$\Rightarrow v(x,t) = \sin\left(\frac{x}{2}\right) e^{-\frac{t}{4}}$$

Full solution:

$$u(x,t) = v(x,t) + u_s(x)$$
$$= \sin\left(\frac{x}{2}\right) e^{-\frac{t}{4}} + \cos(x) + x - 1$$

#

Q3

$$u(x,t)=\sum_{n=-\infty}^{\infty} C_n(t)e^{inx}, \qquad Q(x,t)=\sum_{n=-\infty}^{\infty} q_n(t)e^{inx}$$

$$\downarrow$$

$$q_0(t)=t$$
$$q_1(t)=\frac{1}{2i}\cos(t)$$
$$q_2(t)=\frac{1}{2}$$

$$\Rightarrow \quad \ddot{C}_n=(-n^2+n^4)C_n+q_n$$

Only $n=0$, 1, and 2 matter.

$n=0$: $\ddot{C}_0=q_0=t \Rightarrow \dot{C}_0(t)=\cancel{\dot{C}_0(0)}^{\,0}+\frac{t^2}{2}$

$$C_0(t)=\cancel{C_0(0)}^{\,0}+\frac{t^3}{6}$$

$n=1$: $\ddot{C}_1=q_1=\frac{1}{2i}\cos(t)$

$$\dot{C}_1(t)=\underbrace{\cancel{\dot{C}_1(0)}}_{0}+\frac{1}{2i}\int_0^t\cos(t)\,dt=\frac{1}{2i}\sin(t)$$

$$C_1(t)=\underbrace{\cancel{C_1(0)}}_{0}+\frac{1}{2i}\int_0^t\sin(t)\,dt$$
$$=\frac{i}{2}[\cos(t)-1]$$

From $u(x,0)$ and $u_t(x,0)$ in the b.c.'s:

$$\boxed{\begin{aligned}&C_0(0)=0,\ C_1(0)=0,\ C_2(0)=0\\&\dot{C}_0(0)=0,\ \dot{C}_1(0)=0,\ \dot{C}_2(0)=0\end{aligned}}$$

$n=2$: $\ddot{C}_2=12C_2+\frac{1}{2}$ $\qquad$ Let $D\equiv C_2+\frac{1}{24} \Rightarrow \ddot{D}=12D$

$$D(t)=A\cosh(\sqrt{12}\,t)+B\sinh(\sqrt{12}\,t)$$

$$\Rightarrow \quad C_2(t)=A\cosh(\sqrt{12}\,t)+B\sinh(\sqrt{12}\,t)-\frac{1}{24}$$

From the b.c.: $C_2(0)=0$ and $\dot{C}_2(0)=0 \Rightarrow B=0,\ A=\frac{1}{24}$

Full solution:

$$u(x,t)=C_0(t)+\left\{\left[C_1(t)e^{ix}+C_2(t)e^{i2x}\right]+c.c.\right\}$$

$$=\frac{t^3}{6}+\left\{\left[\frac{i}{2}[\cos(t)-1]e^{ix}+\frac{1}{24}[\cosh(\sqrt{12}\,t)-1]e^{i2x}\right]+c.c.\right\}$$

$$=\frac{t^3}{6}+[1-\cos(t)]\sin(x)+\frac{1}{12}[\cosh(\sqrt{12}\,t)-1]\cos(2x)$$

#

Q4

The steady solution $u_s(x)$ satisfies

$$\begin{cases} u_s'' + u_s = 0 \Rightarrow \underline{u_s(x) = A\cos(x) + B\sin(x)} \\ \underline{u_s'(0) = 1, \quad u_s(2\pi) = 2} \end{cases}$$

$\cancel{u_s}$

$A = 2, \; B = 1$

$$\Rightarrow u_s(x) = 2\cos(x) + \sin(x)$$

Let $v(x,t) \equiv u(x,t) - u_s(x)$

$$\Rightarrow \begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + v \\ v_x(0,t) = 0, \quad v(2\pi, t) = 0, \quad v(x,0) = \cos\left(\frac{x}{4}\right) \end{cases}$$

separation of var. : $v \sim G(x)H(t)$

$$\Rightarrow G'' = cG, \quad G'(0) = 0, \quad G(2\pi) = 0$$

$$\Rightarrow c_n = -\left(\frac{n}{4}\right)^2, \quad G_n(x) = \cos\left(\frac{nx}{4}\right)$$

Only $n=1$ matter.

$$\frac{\dot{H}_1}{H_1} - 1 = -\frac{1}{16} \Rightarrow \dot{H}_1 = \left(\frac{15}{16}\right)H_1 \Rightarrow H_1(t) = \underbrace{H_1(0)}_{\text{set to } 1} e^{\frac{15}{16}t}$$

$$v(x,t) = a_1 G_1(x) H_1(t) = a_1 \cos\left(\frac{x}{4}\right) e^{\frac{15}{16}t}$$

$a_1 = 1$ from b.c.

$$\Rightarrow v(x,t) = \cos\left(\frac{x}{4}\right) e^{\frac{15}{16}t}$$

Full solution:

$$u(x,t) = v(x,t) + u_s(x)$$

$$= \cos\left(\frac{x}{4}\right) e^{\frac{15}{16}t} + 2\cos(x) + \sin(x)$$

#