Q3

By MOC, we have $\frac{dx}{dt} = 0.5 + u$, $\frac{du}{dt} = -u$. They lead to two equations,

$$x = x_0 + 0.5t + p(x_0)(1 - e^{-t}) \quad \text{—— ①}$$

$$u(x,t) = p(x_0)\, e^{-t} \quad \text{—— ②}$$

[Here, we use the shorthand of $x \leftarrow x(t)$, $x_0 \leftarrow x(0)$]

(i) when $x_0 < 0 \Rightarrow p(x_0) = 0 \Rightarrow \boxed{u(x,t) = 0}$

$\Rightarrow x = x_0 + 0.5t$

$\boxed{x < 0.5t}$

(iii) when $x_0 > 1 \Rightarrow p(x_0) = 1 \Rightarrow \boxed{u(x,t) = e^{-t}}$

$\Rightarrow x = x_0 + 0.5t + (1 - e^{-t})$

$\boxed{x > 2 + 0.5t - e^{-t}}$

(ii) when $0 \le x_0 \le 1 \Rightarrow P(x_0) = x_0^2 \Rightarrow u(x,t) = x_0^2 e^{-t}$

$\Downarrow$

$$x = x_0 + 0.5t + x_0^2(1 - e^{-t}) \quad \text{—— (*)}$$

$\boxed{0.5t \le x \le 2 + 0.5t - e^{-t}}$

$$x_0 = \frac{-1 \pm \sqrt{1 - 4(0.5t - x)(1 - e^{-t})}}{2(1 - e^{-t})}$$

[Reject the root with $x_0 = \frac{-1 - \sqrt{\blacktriangle}}{2(1 - e^{-t})}$ since it's negative, violating "$0 \le x_0 \le 1$"]

$$\boxed{u(x,t) = e^{-t}\left\{\frac{-1 + \sqrt{1 - 4(0.5t - x)(1 - e^{-t})}}{2(1 - e^{-t})}\right\}^2}$$

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Note: For Case (ii) From (*), we can write $x - 0.5t = x_0 + x_0^2(1 - e^{-t})$.

Since $0 \le x_0 \le 1$, this means $0 \le x - 0.5t \le 1 + 1 \cdot (1 - e^{-t})$

$\Rightarrow 0.5t \le x \le 2 + 0.5t - e^{-t}$.

Also, since $x - 0.5t \ge 0$, the term inside the $\sqrt{}$ in the formula for x_0 is always positive and greater than 1.

Q4

By MOC, we have

$$\frac{du}{dt} = x \quad \text{— ①} \qquad \frac{dx}{dt} = u \quad \text{—— ②}$$

Combining ① & ② $\Rightarrow \frac{d^2u}{dt^2} = u \Rightarrow u(t) = A\cosh(t) + B\sinh(t)$

From the b.c., $u(0) = 1 \Rightarrow A = 1. \Rightarrow u(t) = \cosh(t) + B\sinh(t)$ — ③

plugging ③ into ①, $x(t) = \frac{du}{dt} = \sinh(t) + B\cosh(t)$

$$\Rightarrow B = \frac{x(t) - \sinh(t)}{\cosh(t)}.$$

$$\Rightarrow u(t) = \cosh(t) + \left[\frac{x(t) - \sinh(t)}{\cosh(t)}\right]\sinh(t)$$

Full solution is $u(x,t) = \cosh(t) + \left[\frac{x - \sinh(t)}{\cosh(t)}\right]\sinh(t).$

Using the relation, $\cosh^2(t) - \sinh^2(t) = 1$, it can be simplified ~~int~~ into

$$u(x,t) = \frac{1 + x\sinh(t)}{\cosh(t)}. \quad \#$$

As $t \to \infty$, $1/\cosh(t) \to 0$ and $\sinh(t)/\cosh(t) \to 1$.

So, the steady solution is $u_s(x) = x$. #

Q5

Using the hint given in class, the original PDE can be decomposed into

$$\frac{\partial w}{\partial t} - \frac{\partial w}{\partial x} = -2w \quad \text{— ①}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = w \quad \text{— ②}$$

The b.c. for ① can be obtained from the original b.c.'s as

$$w(x,0) = u_t(x,0) + u_x(x,0) = x \quad \text{— (1A)}$$

Solving ① & b.c. (1A), we obtain $w(x,t) = (x+t)e^{-2t}$. Therefore, ② becomes

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = (x+t)e^{-2t} \quad \text{— (2*)}$$

Solving (2*) with the original b.c., $u(x,0) = 1$, we obtain

$$u(x,t) = 1 + \frac{1}{2}(x-t)(1-e^{-2t}) + \frac{1}{2}[1 - e^{-2t}(2t+1)]$$

$$= \frac{1}{2}(3+x-t) - \frac{1}{2}(1+x+t)e^{-2t}$$

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